Analytical Solutions of the Electrostatically Actuated Curled Beam Problem

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ABSTRACT

We present analytical solutions of the electrostatically actuated initially deformed cantilever beam problem. We use a continuous Euler-Bernoulli beam model combined with a single-mode Galerkin approximation. We derive simple analytical expressions for two commonly observed deformed beams configurations: the curled and tilted configurations. The derived analytical formulas are validated by comparing their results to experimental data in the literature and numerical results of a multi-mode reduced order model. The derived expressions do not involve any complicated integrals or complex terms and can be conveniently used by designers for quick, yet accurate, estimations. The formulas are found to yield accurate results for most commonly encountered microbeams of initial tip deflections of few microns. For largely deformed beams, we found that these formulas yield less accurate results due to the limitations of the single-mode approximations they are based on. In such cases, multi-mode reduced order models need to be utilized.

Keywords: curled, cantilever, microbeam, electrostatic, pull-in

1 INTRODUCTION

Cantilever microbeams are commonly fabricated with unavoidable initial tilt or curling due to stress gradients and other imperfections [1-3]. Despite the low level of this initial deflection of the beam profile compared to its length; it has significant effect on its static and dynamic behavior. This is even more critical in the case of electrostatic excitation and capacitive detection, which have strong dependence on the gap separating the cantilever beam from the substrate or lower electrode.

Several attempts have been made to develop analytical formula to predict the pull-in voltage of curled microbeams [4-7]. However, these mostly have resulted in cumbersome expressions, which involve complicated terms and integrals rendering them unusable and less practical.

It would be of great advantage to MEMS designers to be able to use simple formula that can be programmed on calculators and that do not require intensive calculations. Such analytical expressions can be considered handy, pure analytical, and practical. In this paper, we aim to present such formulas. Starting from a continuous Euler-Bernoulli beam model combined with a single-mode Galerkin approximation, we solve the bifurcation problem and derive simple analytical expressions for two commonly observed deformed beams configurations: the curled and tilted configurations.

2 PROBLEM FORMULATION

We consider an electrostatically actuated cantilever microbeam of length l, thickness h, and width b. The beam anchor is raised above the substrate a distance d while the beam itself is curled above the anchor level with a profile $\tilde{g}(\hat{x})$, where $\hat{x}$ is the position along the beam length, as shown in Fig. 1.

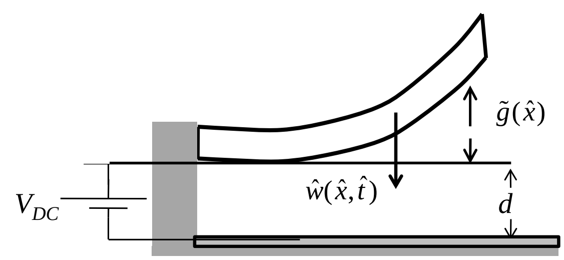


Figure 1. Schematic of an initially deformed electrostatically actuated cantilever microbeam.

The equation of motion governing the deflection of the microbeam in space $\hat{x}$ and time $\hat{t}$ can be written as

$$EI\hat{w}'''' + c\dot{\hat{w}} + \rho bh\ddot{\hat{w}} = \frac{\varepsilon b V_{DC}^2}{2\left(d - \hat{w} + \tilde{g}\right)^2} \tag{1}$$

For convenience, Eq. (1) is normalized. Toward this, the following nondimensional variables (denoted by hats) are introduced

$$w = \frac{\hat{w}}{d}, \qquad x = \frac{\hat{x}}{l}, \qquad t = \frac{\hat{t}}{T} \tag{2}$$

where T is a time scale defined as $T = \sqrt{\rho bhl^4 / EI}$.

Substituting Eq. (2) into Eq. (1), the following nondimensional equation is derived:

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} + c_{non}\frac{\partial w}{\partial t} = \frac{\alpha_2 V_{DC}^2}{(1 - w + g)^2} \tag{3}$$

where $g(x)$ is the nondimensional beam profile. The parameters appearing in Eq. (3) are defined as

$$\alpha_2 = \frac{6\varepsilon l^4}{Eh^3 d^3}; \quad c_{non} = \frac{12cl^4}{ETbh^3}; \tag{4}$$

Starting from Eq. (3) combined with a single-mode Galerkin approximation [3, 8], we solve the bifurcation problem and derive simple analytical expressions for two commonly observed deformed beams configurations: the curled and tilted configurations.

Cantilever microbeams can be initially curled due to stress gradient, which is equivalent to an applied moment on the beam, or can be tilted due to the flexible anchor [1-3]. The stress gradient results in a parabolic shape of the beam, which can be expressed as

$$g(x) = \gamma x^2 \tag{5}$$

where $\gamma = W_0^{tip}/d$, where W_0^{tip} is the maximum tip deflection of the cantilever beam, which can be measured using an optical profiler or interferometer.

In the case of a beam titled due to non-ideal support, the profile looks like a linear one rather than a curvy parabolic

$$g(x) = \gamma x \tag{6}$$

where also here $\gamma = W_0^{tip}/d$.

Next, we use the first undamped linear mode shape of the unactuated microbeam as a basis functions in the Galerkin procedure. To this end, we express the deflection as

$$w(x,t) = u_1(t)\phi_1(x) \tag{7}$$

where $u_1(t)$ is the generalized coordinate and $\varphi_1(x)$ is the first linear undamped mode shape of the microbeam. Applying the Galerkin pricedure, we obtain

$$\left[\ddot{u}_1 + c_{non}\dot{u}_1 + \omega_{non}^2 u_1\right]\times \int_0^1 \phi_1\left\{\left(\phi_1 + 2g\phi_1 + g^2\phi_1 - 2u_1\phi_1^2 - 2gu_1\phi_1^2 + u_1^2\phi_1^3\right)\right\}dx = \alpha_2 V_{DC}^2 \int_0^1 \phi_1 dx \tag{8}$$

Finally, using the expressions of Eq. (5) and (6) and then applying a bifurcation analysis, which requires the eigevalue of the linearized system equation of Eq. (8) to vanish at pull-in [3], yeild the below expressions for W_p, which is the maximum tip deflection of the cantilever at pull-in, and V_{pull}, which is the corresponding voltage.

Curled:

$$W_p = 0.84 + 0.62\gamma - 0.31\sqrt{(0.65+\gamma)(2.23+\gamma)} \tag{9}$$

$$V_{pull} = \sqrt{\left[7.9W_p\ (1. + 0.59\ W_p^2\ + W_p\ (-1.48 - 1.1\ \gamma) + 1.35\ \gamma + 0.51\gamma^2)\right]/\alpha_2} \tag{10}$$

Titled:

$$W_P = 0.84 + 0.71\gamma - 0.35\sqrt{(0.71+\gamma)(1.53+\gamma)} \tag{11}$$

$$V_{Pull} = \sqrt{\left[7.9W_p\ (1. + 0.59\ W_p^2\ + W_p\ (-1.48 - 1.26\ \gamma) + 1.61\ \gamma + 0.675\gamma^2)\right]/\alpha_2} \tag{12}$$

3 RESULTS

In this section, the validity of the derived formulas of Eq.(10) and Eq.(12) is demonstrated by comparison with the experimental data of [7]. These data represent a comprehensive experimental study of initially deformed cantilever beams of lengths ranging from 100-500 μm. They are commonly used in the literature for comparison purposes with the modeling results. Table 1 below lists the material and geometric properties of the beams array. Note here that all the beams share the same stress gradient, and hence, the same radius of curvature *R*. Table 2 lists the beam lengths and their maximum tip deflection, as calculated based on the provided radius of curvature ($W_0^{tip} = R(1-\cos(l/R))$, in addition to their measured pull-in voltage [7].

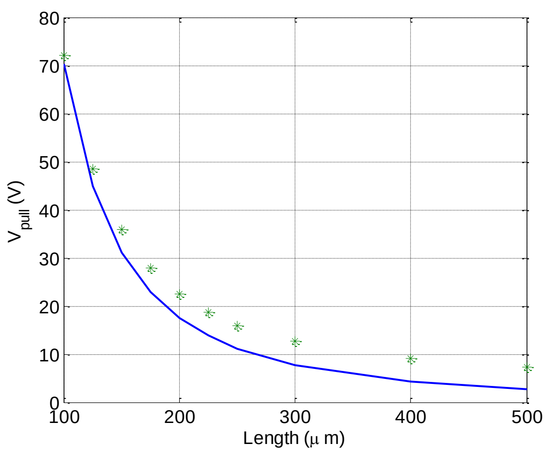
Parameter	**Value**
Young's modulus, *E*	153 GPa
Relative dielectric constant between the beam and the substrate ε_r	1.2046
Beam length, *L*	100–500 μm
Beam width, *b*	40 μm
Beam thickness, *h*	2.1 μm
Initial radius of curvature, *R*	40 000 μm
Initial gap, *d*	2.4 μm

Table 1. Material and geometrical parameters of the polysilicon curled beams of [7].

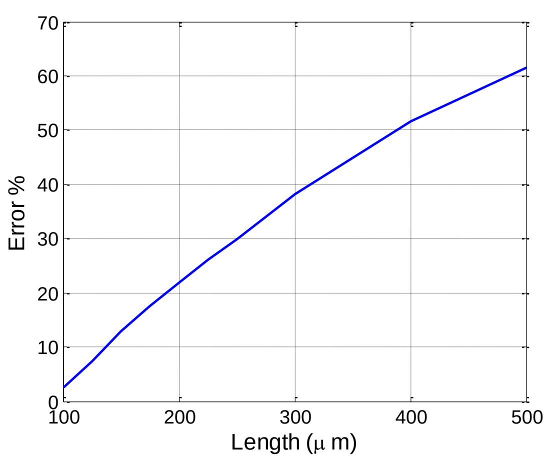
Before showing results using the new model of the curved beams, it is worth to compare the reported experimental data of [7] with the theoretical results assuming a straight beam, and using a beam model. Figure 2a depicts the comparison whereas Fig. 2b shows the estimated error. For small beams the initial deflection is too small to make any difference, but as the beam length increases, the initial deflection increases, and so does the error. It is clear that using a straight beam theory for pull-in calculations of initially deformed beams leads to severely wrong results.

Length (μm)	W_0^{tip} (μm)	Measured Pull-in [7]
100	0.125	72.07
125	0.195	48.6
150	0.28	35.82
175	0.38	27.89
200	0.5	22.55
225	0.63	18.79
250	0.78	15.95
300	1.12	12.61
400	2	9.10
500	3.12	7.27

Table 2. The length of the microbeams of [7], their maximum tip deflection, and their measured pull-in voltages.



(a)



(b)

Figure 2. (a) Pull-in voltage of a curled beam calculated using a straight beam theory (solid) versus the experimental data of [7]. (b) The calculated percentage error.

Next, we apply both the curled and tilted formulas, Eq.(10) and Eq.(12), on the beams of Tables 1 and 2 to determine their pull-in voltage. The results as compared to the experimental measurements of [7] are depicted in Fig.3. As seen, both formulas predict results that are very close to the experimental data. In addition, we calculated the pull-in of these beams using a 4-mode reduced order model ROM [3,8] when implementing both the curled and tilted configurations. The results of the ROM, the analytical formulas, and the experimental data all show excellent agreement. The multi-mode ROM in this particular set of data has not shown much improvement in accuracy. The tilted and curled model here show close results. However this cannot be known for sure unless accurate profiles of the beams are captured using interferometers or SEM pictures. In general it is expected that, in the case of uncertain initial deformation profile, both formulas will capture the proper range of the actual data.

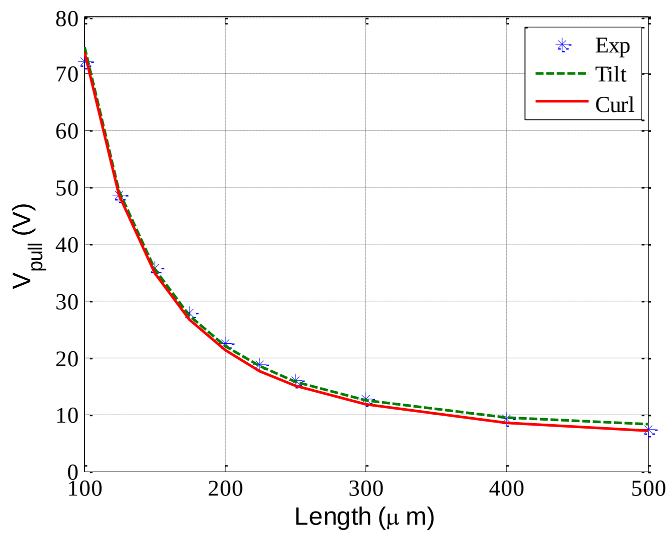


Figure 3. Comparison of the experimental data of [7] against the theoretical predictions using the tilted and curled beam models.

4 LIMITATIONS

Despite the excellent accuracy of the developed formulas demonstrated in the previous section, one should recall that these are based on a single-mode approximation. Therefore, it is expected that these formulas are limited in their accuracy for small initial deformations, as in the case of the beams of Table 2. For instance, it was found previously that multiple modes are needed to capture accurately the pull-in instability under mechanical shock and that a single mode prediction is inaccurate [9]. Based on this, we examine the limitation of the developed formulas by studying largely initially deflected beams. As case studies, we investigate the beams of Hu and Wei [6] of properties listed in Table 4.

Parameter	Value
Young's modulus, *E* (GPa)	74.14
Beam length, *l* (μm)	400, 500, 600
Initial radius of curvature, ρ (μm)	2781, 3641, 4359
Beam width, *b* (μm)	50
Beam thickness, *h* (μm)	1.32
Initial gap, *d* (μm)	1.63
Relative dielectric constant between the beam and the ground	1

Table 3: The material and geometrical parameters of the curled aluminum beam of [6].

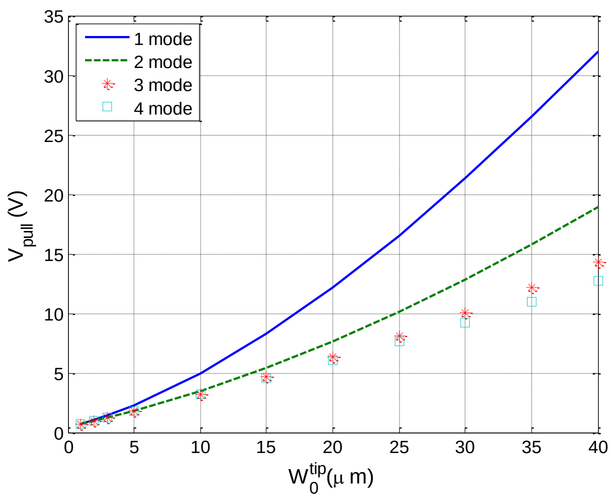


Figure 4. Comparison of the pull-in voltage as calculated using 1-4 modes of the ROM.

Here, we adopt the 600 μm of Table 3, as a case study, and investigate its pull-in voltage as calculated using 1-4 modes of the ROM, while its maximum tip deflection varies from very small values to large values, Fig. 4. The figure indicates that using 3 and 4 modes yields close and converged results for most of the tip-deflection range. Using the analytical formula, or the 1-mode ROM, is accurate only up to a maximum tip deflection of 5 μm. Hence, this establishes the range of validity of the developed analytical formulas.

5 SUMMARY

We presented analytical solutions of the electrostatically actuated initially deformed cantilever beam problem. We used a continuous beam model and a single-mode Galerkin technique. We derived simple analytical expressions for two commonly observed deformed beams configurations: the curled and tilted configurations. We compared the results of the derived formulas to experimental results in the literature and numerical results of a multi-mode reduced order model. We found that these formulas yield accurate results for beams of tip deflections of few microns, which is the case commonly encountered in MEMS applications. For largely deformed beams, these formulas cannot be used due to the limitations of the single-mode approximations they are based on. Instead, multi-mode reduced order models need to be utilized. For choosing the curled or titled formulas, one should resort to optical images of the fabricated beams; otherwise formulas both can be used to catch the expected range of pull-in.

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NSTI-Nanotech 2014, www.nsti.org, ISBN 978-1-4822-5827-1 Vol. 2, 2014