A Sparse Grid based Collocation Method for Model Order Reduction of Finite Element Models of MEMS under Uncertainty

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ABSTRACT

A methodology is proposed for the model order reduction of finite element approximations of MEMS devices under random input conditions. In this approach, the reduced order system matrices are represented in terms of their convergent orthogonal polynomial expansions of input random variables. The coefficients of these polynomials, which are matrices, are obtained by repeated, deterministic model order reduction of finite element models generated for specific values of the input random variables. These values are chosen efficiently in a multi-dimensional grid using a Smolyak algorithm. The stochastic reduced order model is represented in the form of an augmented system which can be used for generating the desired statistics of the specific system response. The proposed method provides for significant improvement in computational efficiency over standard Monte Carlo.

Keywords: stochastic, model order reduction, mems, random, finite element

1 INTRODUCTION

Model Order Reduction (MOR) has facilitated the development of efficient computational tools for MEMS analysis and design [1]. However, in their majority, reduced order modeling tools assume a deterministic definition of the structure being modeled. Variations and/or uncertainty in geometric attributes, material properties and operating conditions are becoming a significant impacting factor in MEMS device performance. These need to be accounted for in model order reduction for better design and optimization. This has led to a lot of research in the development of variational and parametric models for analysis of MEMS devices. In this paper, we put forward a framework for performing model order reduction in the presence of uncertainty.

There are broadly two different approaches for handling uncertainty, statistical and non-statistical. In terms of statistical techniques, brute-force Monte Carlo computation [2] is the most straightforward approach. In such a method, one considers a large number (typically greater than 10000) realizations (or samples) of the model and a deterministic problem is solved for each one of these realizations. The data is then used for the development of the statistics of the desired observable quantity. Clearly, such an approach is time consuming, with a convergence rate in obtaining reliable statistics of $O(N^{-1/2})$, where N is the number of samples.

In terms of non-statistical approaches, polynomial chaos, a spectral expansion of the stochastic processes in terms of orthogonal polynomials has been extensively used [3], [4]. In the context of model order reduction, since conventional polynomial chaos requires significant new code development, its utilization in conjunction with existing MOR technology is hindered by the overhead of new algorithm development. An answer to this difficulty is the stochastic collocation method [5], which combines the advantages of both stochastic Galerkin methods and classical Monte Carlo approaches. The key idea is to use a "decoupled" polynomial interpolation in multi-dimensional random space. Thus, all that is required is a run of the deterministic solver for each point in the multi-dimensional parameter space. These points can be generated using various algorithms, such as Smolyak sparse grids. Stochastic collocation is also more efficient than brute force Monte Carlo due to the smart choice of interpolation points. In [6], Xiu described the use of Smolyak sparse grids for parametric uncertainty analysis.

In this paper, our focus is to develop a method that treats MOR as a black-box deterministic solver. In other words, the objective is for the approach to be independent of the specific MOR algorithm employed. For the purpose of illustration, we make use of the Krylov subspace MOR algorithm.

2 THEORY

2.1 Deterministic Reduced Order Model

The dynamic behavior of a MEMS device can be described through a coupled electro-mechanical model which is represented in the form of following second-order system of equations

$$M\ddot{x}(t) + D\dot{x} + Kx = F_{elec}(x, t) \quad (1)$$

where N is the number of degrees of freedom in the finite element approximation, the vector x is the displacement of the movable electrode, the matrices M, D, K are in $R^{N\times N}$. $F_{elec}(x,t)$ represents the electrostatic force of attraction acting on the movable electrode. Note that it is non-linearly dependent on the state vector x. In order to describe the model order reduction methodology, we adopt a terminology consistent with the one used in [1]. The matrices M, D, K are assumed to be time-invariant. Henceforth, we use the subscript *org* for indicating the full-order finite element model, Σ_N. Thus, we have,

$$\begin{aligned} M_{org}\ddot{x}(t) + D_{org}\dot{x} + K_{org}x &= b_{org}u(t) \\ y(t) &= l^t_{org}x(t) \end{aligned} \tag{2}$$

where $b_{org}, l_{org} \in R^N$ are input distribution and output measurement vectors respectively.

A Second Order Arnoldi (SOAR) method is used for performing MOR [1]. In this approach, a transformation matrix Q in $R^{N\times n}$ is developed where $n << N$. n determines the size of the reduced order system, Σ_n. The reduced order system can then be computed as,

$$\begin{aligned} M\ddot{z}(t) + D\dot{z} + Kz &= bu(t) \\ y(t) &= l^T z(t) \end{aligned} \tag{3}$$

where,

$$\begin{aligned} K = Q^T K_{org}Q, \quad D = Q^T D_{org}Q \quad M = Q^T M_{org}Q \\ b = Q^T b_{org} \quad l = Q^T l_{org} \end{aligned} \tag{4}$$

The K, D, M matrices are now in $R^{n\times n}$.

2.2 Stochastic Reduced Order Model

Next, it is assumed that the material and/or the geometric properties of the structure exhibit uncertainty. This uncertainty leads to randomness in the description of the system matrices. Let the stochastic reduced order system $\tilde{\Sigma}_n$ be represented by,

$$\begin{aligned} \tilde{M}\ddot{\tilde{z}}(t) + \tilde{D}\dot{\tilde{z}} + \tilde{K}\tilde{z} &= \tilde{b}u(t) \\ \tilde{y}(t) &= l^T\tilde{z}(t) \end{aligned} \tag{5}$$

where the superscript $\sim$ is used to indicate randomness. Thus, it is,

$$\begin{aligned} \tilde{K} = \tilde{Q}^T\tilde{K}_{org}\tilde{Q}, \tilde{D} = \tilde{Q}^T\tilde{D}_{org}\tilde{Q}, \tilde{M} = \tilde{Q}^T\tilde{M}_{org}\tilde{Q} \\ \tilde{l} = \tilde{Q}^T\tilde{l}_{org}, \tilde{b} = \tilde{Q}^T\tilde{b}_{org} \end{aligned}$$

The next step involves the representation of randomness in the original system matrices. This is done by making use of polynomial chaos expansion [3], which is an expansion in terms of orthogonal polynomials of random variables. For the purpose of illustration and brevity, two input variables are assumed to be random. Considering a linear expansion in both of the input random variables, we have,

$$\begin{aligned} \tilde{K}_{org} = K_0 + K_1\xi_1 + K_2\xi_2, \tilde{D}_{org} = D_0 + D_1\xi_1 + D_2\xi_2 \\ \tilde{M}_{org} = M_0 + M_1\xi_1 + M_2\xi_2, \tilde{Q} = Q_0 + Q_1\xi_1 + Q_2\xi_2 \end{aligned} \tag{6}$$

Note that in the above expansions the coefficients (such as K_0,K_1) are deterministic matrices, while $1, \xi_1, \xi_2$ are orthogonal polynomials in the two-dimensional random space [3]. Once the coefficient matrices have been found, the expansions above can be computed to evaluate any system matrix corresponding to a particular value of the input random variables. One way of computing the coefficients is by integrating both sides over two-dimensional random space. For example,

$$\int \tilde{K}_{org}\rho_1\rho_2 d\xi_1 d\xi_2 = \int (K_0 + K_1\xi_1 + K_2\xi_2)\rho_1\rho_2 d\xi_1 d\xi_2$$

$$K_0 = \int \tilde{K}_{org}\rho_1\rho_2 d\xi_1 d\xi_2 \tag{7}$$

where ρ_1,ρ_2 are probability density functions (pdf) corresponding to the random variables ξ_1,ξ_2, respectively. Let $I(f)$ denote the integrals in (7), where

$$f = \tilde{K}_{org}\rho_1(\xi_1)\rho_2(\xi_2).$$

The accuracy of finding these coefficients depends on the accuracy in the calculation of these integrals. Their efficient and accurate calculation is discussed in the next subsection.

Once these coefficients are obtained, an augmented reduced order system can be defined. Using (5) and (6),

$$\begin{bmatrix} M_0 & M_1 & M_2 \\ M_1 & M_0 & 0 \\ M_2 & 0 & M_0 \end{bmatrix} \begin{bmatrix} \ddot{z}_0 \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} D_0 & D_1 & D_2 \\ D_1 & D_0 & 0 \\ D_2 & 0 & D_0 \end{bmatrix} \begin{bmatrix} \dot{z}_0 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} K_0 & K_1 & K_2 \\ K_1 & K_0 & 0 \\ K_2 & 0 & K_0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} u(t)$$

The above system may be cast in compact matrix form as,

$$M_{aug}\ddot{z}_{aug} + D_{aug}\dot{z}_{aug} + K_{aug}z_{aug} = b_{aug}u(t) \tag{8}$$

The augmented system matrices $M_{aug}, D_{aug}, K_{aug}$ are $3n \times 3n$.

2.3 Numerical Integration

In the above development the accurate and efficient computation of the coefficient stiffness matrices using the integrals of the type in (7) is of significant importance. There exist several well-established rules for the case of one-dimensional (1-d) integration (e.g., the use of Chebyshev polynomials). The challenge lies in extending these rules to multiple dimensions efficiently. Consider an N-dimensional random space. Let $I^{q_i}_i$, $i = 1, 2, .., N$, denote the 1-d integration rule in the ith direction, involving q_i points. Thus

$$I^{q_i}_i[f] = \sum_{j=1}^{q_i} f(u^j_i).w^j_i \tag{9}$$

based on nodal sets

$$\Theta_i^1 = (u_i^1, ..., u_i^{q_i}) \subset \Gamma_i \quad (10)$$

A straightforward approach would be to use a Tensor product. However, the number of points in this approach grows rapidly as q^N.

$$I^Q[f] \equiv (I_i^{q_1} \otimes ... \otimes I_i^{q_N})[f]$$
$$= \sum_{j_1=1}^{q_1} ... \sum_{j_N=1}^{q_N} f(u_1^{j_1}, ..., u_N^{j_N}).(w_1^{j_1} \otimes ... \otimes w_N^{j_N}) \quad (11)$$

An alternative and more efficient approach makes use of Smolyak sparse grids. The Smolyak algorithm is a smart linear combination of product formulas chosen in such a way that an integration property that holds for $N = 1$ is preserved as accurately as possible for the case of $N > 1$. Only products with a relatively small number of points are used and the resulting nodal set has significantly less number of nodes compared to those generated by the tensor product rule. Details of the Smolyak algorithm can be found in [7].

The Smolyak algorithm is given by

$$I^Q(f) \equiv$$
$$\sum_{J-N+1 \le |i| \le J} (-1)^{J-|i|} \cdot \begin{pmatrix} N-1 \\ J-|i| \end{pmatrix} \cdot (I_{i_1} \otimes .. \otimes I_{i_N})$$

where $i = (i_1, ..., i_N) \in N^N$. To compute $I^Q(f)$ we only need to evaluate the function on the "sparse grid"

$$\Theta_N \equiv H(J, N) = \bigcup_{J-N+1 \le |i| \le J} (\Theta_{i_1}^1 \times ... \times \Theta_{i_N}^1) \quad (12)$$

If P denotes the total degree of the multi-variate orthogonal polynomial, then it can be shown that the number of points in a Smolyak grid varies as [6],

$$Q \sim \frac{2^P}{P!} N^P, \qquad P \text{ fixed.} \quad (13)$$

Clearly, the dependence on dimension N is much weaker than the tensor product rule. The number of points in the Smolyak sparse grid are much smaller compared to those in Tensor product grid for equivalent accuracy.

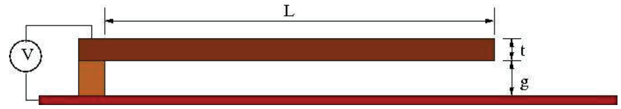
2.4 Algorithm for Stochastic MOR

With the pertinent mathematical framework in place the proposed algorithm for stochastic model order reduction is as follows:

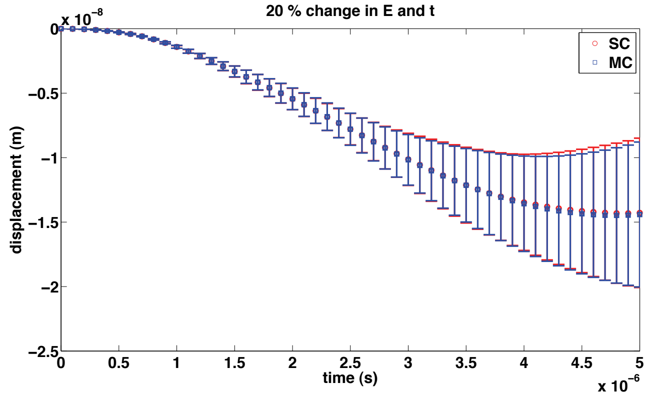
- Definition of the number and type of distribution of input random variables.
- Choose an appropriate order and type of polynomial chaos expansion (eqn.(6)).
- Generate a Smolyak sparse grid. Each point represents a combination of values for different input random variables (eqn. (11)).
- For each point on the sparse grid, calculate the full-finite element system matrices. Perform deterministic MOR for each system to generate the corresponding transformation matrix (eqns. (3) and (4)).
- Using above information, calculate the coefficients in the polynomial expansion (eqn. (7)).
- Compute the augmented system using the coefficients (eqn. (8)).
- The augmented system can be used for calculating the mean, standard deviation and other statistics of the system response.

3 Numerical Study - Cantilever switch

In this section, we consider a cantilever switch (Fig. 1(a)) for demonstrating the application of our proposed algorithm for stochastic MOR. The top electrode is 80 μm long, 0.5 μm thick and 10 μm wide and is suspended 0.7 μm over the bottom electrode. A Youngs modulus of 169 GPa, a mass density of 2231 kg/m and a Poissons ratio of 0.3 is assumed. No damping is considered. A full-finite element model consisting of 100 elements is constructed using Euler-Bernoulli beam theory. A reduced order model of order 5 is employed. A step voltage of 0.5 V is used. The output response of interest is the peak displacement, which occurs at the tip of the cantilever beam. A time step of 0.1 s was used in the transient simulations. We consider random variations in two input parameters - the Young's modulus E and the thickness of the beam t, both with uniform distribution. Note that these parameters affect the stiffness as well as the mass matrices. We consider 10% and 20% variations in these parameters, and look at their impact on the transient behavior of the switch. For reference, we use standard Monte Carlo (MC) simulations. Thus we consider 10,000 realizations of the input parameters E and t, generate the full-finite element model and the reduced order model for each sample and then perform the transient simulation using the reduced order model. Responses for all the samples are collected to generate statistics such as mean and standard deviation. For our approach (SC), we make use of level 4 Smolyak algorithm consisting of only 29 grid points in the two dimensional random parameter space for numerical integration. The results are summarized in Figure 1(b). Figure 1(b) shows a comparison of the mean displacement with error bars corresponding to one standard deviation obtained using Monte Carlo and our approach. Very good agreement is observed. A numerical comparison is given in Table 1. It shows results for mean



(a)



(b)

Figure 1: (a) Cantilever Switch (b) Stochastic tip displacement for 20% change in 'E' and 't'

Table 1: Cantilever switch (Maximum displacement (m)): Statistical change in Young's modulus 'E' and thickness 't'

%change	Monte Carlo		Proposed method	
	mean	std. dev	mean	std. dev
10%	-1.32e-8	3.07e-9	-1.35e-8	2.88e-9
20%	-1.44e-8	5.62e-9	-1.43e-8	5.79e-9

and standard deviation of displacement at the last time step. It is clear that there is a good match between results obtained using MC and our approach.

4 Summary

In summary, we have presented a systematic methodology for the model order reduction of finite element models of MEMS structures exhibiting statistical variability in their material and geometry parameters. The proposed methodology makes use of stochastic collocation and standard Krylov model order reduction techniques to develop a stochastic reduced model of order small enough to enable efficient quantitative assessment under input uncertainty. Furthermore, the methodology is independent of the MOR algorithm used. Thus, it is seamlessly compatible with MOR toolkits used in popular finite element solvers.

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