# Linear Analysis of Large Deflection of Heated and Clamped Isotropic Layered Plate Under Initial Tension

C.-F. Chen* and J.-C. Yu**

* Institute of Mechanical and Aeronautical Engineering, Chung-Hua University
No. 707, Sec 2, Wu-Fu Rd., Hsin-Chu, Taiwan 30067, R. O. C, cfchen@chu.edu.tw
**Institute of Mechanical and Aeronautical Engineering, Chung-Hua University
No. 707, Sec 2, Wu-Fu Rd., Hsin-Chu, Taiwan 30067, R. O. C

## ABSTRACT

A simplified linear problem is studied for the large deflection of a clamped and heated isotropic circular layered plate under pretension and uniform loading. Including a thermal consideration, governing equations based on von Karman large deflection theory are derived first, followed by neglecting the arising nonlinear term. The non- dimensional structural responses are found to be expressible in terms of modified Bessel functions similar to which available in literature but with a modified definition for the associated arguments. The presented approach is checked against a simplified case for a quasi-monolithic plate. Parametric studies including the effects of pretension, temperature, and ratio between Young's moduli of the layers upon both the global and edge zone behavior of the plate are thoroughly explored. Compared to the previous study for a single-layered plate, visible effect due to the modulus' ratio of the plate is detected, for the considered symmetric layered plate in a thermally elevated condition.

**Keywords**：Layered isotropic plate, Initial tension, Edge Zone effect, von-Karman Plate theory, Modified Bessel functions.

## 1. INTRODUCTION

Pretension which often encountered in a typical micro-fabrication process has been recognized that it could be high enough to cause a drastic degradation in structural performance such as the deflection- based pressure sensitivity [1-2]. For miniaturized structures such as Poly-silicon-based pressure sensors and accelerometers, the situation may be exasperated due to the thermal effect in a high-temperature environment. Since a layered configuration for such a device is very common, understanding of the coupled effects due to pretension, material heterogeneity, and elevated temperature upon the relevant structural responses is very important. Particularly, it may be worthy of note about the onset of nonlinearity of the structural behavior of a sensor plate undergoing a large deflection condition [3-4].

The earliest study in this category seemed to be due to Voorthuyzen & Bergveld [5] that considered the influence of pretension on the deflection of a circular sensing diaphragm but provided solutions for several certain pretensions and lateral pressures only. Allen [6] investigated the influence of in-plane loading upon the central deflection of a polyimide film via an energy method without considering the coupled effect between the surface pressure and pretension. Although Lee & Wise [7] could be the first to study thermal effect-induced lateral stresses on a pressure sensor, they dealt with small deflection condition only. This study is aimed at the simplified linear case for large deflection of a clamped and heated layered plate under uniform tension and lateral pressure. Sheplak and Dugundji's approach [3] will be extended to include thermal consideration in formulating the problem.

## 2. PHYSICAL PROBLEM AND SOLUTION

A symmetrically layered circular plate clamped all around is considered. It is subjected to a uniform pretension, $N_0$, and a uniform lateral pressure , $P_0$, as shown (Fig. 1 and 2). Rather than the non-dimensional slope and radial force resultant employed in Ref. [3], the coupled nonlinear governing equations following force and moment equilibrium for the layered plate are to be expressed, first, in terms of deflection, w, radial displacement, u, including a radial thermal force term. They can further be formulated to take a dimensionless form to read,

$$\begin{cases} \xi^2 U'' + \xi U' - U = -\dfrac{\xi}{\lambda}\left[\xi\theta' + (1-R_{v1})\dfrac{\theta}{2}\right]\theta \\ \xi^2\theta'' + \xi\theta' - \theta - \xi\left\{\xi R_{v2}\left[\dfrac{U'}{\lambda} + \dfrac{\theta^2}{2\lambda^2}\right] + R_{v3}\dfrac{U}{\lambda} - \xi R_{vT} + \xi k^2\right\}\theta = \dfrac{1}{2}\cdot\xi^3 P \end{cases}$$

where $\xi = r/a$，$(\ )' = d/d\xi$，$U = u/h$, $W = w/h$;

$\theta = W,_{\xi} = a/h \cdot w,_{r}$; $\psi = \theta,_{\xi} = a^2/h \cdot w,_{rr}$，$\lambda = \dfrac{a}{h}$;

$U$, $W$, $\theta$, and $\psi$ are the non-dimensional radial displacement, lateral deflection, lateral slope and curvature respectively; and $\xi$, $P$ and $k$ are the non-dimensional radial coordinate, lateral pressure, and tension parameter individually. Among them, the dimensionless pressure ($P$) and pretension parameter ($k$) are defined such that,

$$P=\frac{p_0a^4}{D_l h}, \quad k=\sqrt{\frac{N_0a^2}{D_l}}$$

For the case of small deflection as tension parameter, $k$, varies from 0 to infinity, all the nonlinear terms arisen in the second equation can then be neglected, yielding a non-homogeneous linear differential equation for the non-dimensional slope, i. e.,

$$\xi^2\theta''+\xi\theta'-\theta-\xi^2\left(-R_{vT}+k^2\right)\theta=\frac{1}{2}\cdot\xi^3P$$

This equation can be rewritten in a standard form of modified Bessel equation provided a modified pretension parameter, $k_T$, is employed, i. e.,

$$\xi^2\theta''+\xi\theta'-[1+k_T^2\xi^2]\theta=\frac{1}{2}\cdot\xi^3P;$$

$$k_T=\sqrt{k^2-R_{vT}}$$

In the above equations, $R_{vT}$ s and $R_{vi}s$ (***i***=1 ~ 4) are expressible in terms of the layer rigidities and thermal force resultant. The homogeneous part of solution for the fore-going equation are expressible in terms of modified Bessel functions rendering the following complete solution form for the lateral slope,

$$\theta(\xi)=C_1I_1(k_T\xi)+C_2K_1(k_T\xi)-\frac{P\xi}{k_T^2}$$

where $I_1$ and $K_1$ are the modified Bessel functions of the first kind and the second kind, respectively. The unknown constants, $C_1$ and $C_2$, are to be determined by considering the boundary conditions of the problem that include the clamped condition along the edge and no slope at the center of the plate. Implementing the boundary conditions and considering the limiting case for $K_1$ as $\xi$. → 0, it is seen that, $C_1=P/k_T^2I_1(k_T)$ ; $C_2=0$. The non-dimensional slope, curvature, and deflection can then be obtained to read,

$$\theta(\xi)=P\cdot\left[\frac{I_1(k_T\xi)}{k_T^2I_1(k_T)}-\frac{\xi}{k_T^2}\right]$$

$$\psi(\xi)=P\cdot\left[\frac{I_0(k_T\xi)}{k_TI_1(k_T)}-\frac{1}{k_T^2\xi}\frac{I_1(k_T\xi)}{I_1(k_T)}-\frac{1}{k_T^2}\right]$$

$$W(\xi)=\int_1^{\xi}\theta(\xi)d\xi=\frac{P}{k_T^2}\left[\frac{I_0(k_T\xi)-I_0(k_T)}{k_TI_1(k_T)}-\frac{1}{2}(\xi^2-1)\right]$$

For the limited cases of a pure plate ($k_T$ → 0) and a pure membrane ($k_T$ → ∞), much simpler solution can be obtained, respectively, to read

Pure Plate : $\theta(\xi)=-\frac{1}{8}P\xi(1-\xi^2)$ , $W(\xi)=\frac{1}{32}P(1-\xi^2)^2$

Pure membrane : $\theta(\xi)=-\frac{P\xi}{k_T^2}$ , $W(\xi)=\frac{P}{2k_T^2}(1-\xi^2)$

## 3. NUMERICAL REMARKS

For demonstration, a symmetric three-layered isotropic plate is considered with ratios between layer moduli to be $B_1=(E_1=E_3)/E_2$ =1.05, 1.2, and 1.5. The elevated uniform temperature of the plate is taken to be $T$=25, 50, and 150 (°C) simulating an ordinary ambient and a severe thermally-elevated application conditions. In additions, the range of the tension parameter, $k$, used in Ref. [3] is similarly adopted. The solutions include non-dimensional central deflections, $W_0/P$ versus pretension, $k$, non-dimensional lateral deflections (Fig. 3), slopes (Fig. 4) and curvatures for various layered plates under different temperatures.

### 3.1 Solutions for Central Deflection

Central deflections for various pretensions and modulus ratios ( $B_1$ ) are presented in Figure 2. Regardless of the temperature ($T$), the solutions for $W_0/P$ versus $k$ is found to be close to those of Sheplak and Dugundji [3] and which for low temperature ($T=25^\circ C$) and quasi-monolithic case( $B_1$ =1.05) is even found to agree very well with Sheplak and Dugundji [3] 。 $W_0/P$ is almost a constant in the range of low pretension. As the pretension increases, however, $W_0/P$ varies nonlinearly with $k$ with a nearly square-inverse proportion ($W_0/P\propto 1/k^2$), similar to Sheplak and Dugundji [3].

By comparing the results of Figure 2 and 3, it is visualized that, the effect of modulus ratio becomes apparent only when a moderate or even lower pretension ( $k\ \le 10$ ) is considered. For a relatively high intial tension, the curves merge to the region of nonlinear proportion, indicating that it is less dominant in compared to the tension effect.

In a relatively high temperature condition (Figure 3), in additions, the validity of the previously mentioned nonlinear proportion is found to be narrowed. This implies a comparatively high temperature may reduce the pretension effect of the plate. Correspondingly, the thermal effect is seen only in the condition of high pretension as well.

### 3.2 Results of Global Deflection

The Radial distributions of the lateral deflections for the room temperature ($T=25^\circ C$) and a highly-elevated case ($T=150^\circ C$) is given in Figure 4 and 5 respectively. For the former case ($T=25^\circ C$), the influence due to modulus ratio on the deflection is seen to be reduced as the pretension proceeds. The opposite condition is observed, however, for the latter case ($T=150^\circ C$). This can be attributed to the jointed effect between the modulus ratio, pretension and thermal influence. As a consequence, the effect of modulus ratio tends to be even more apparent in a relatively high temperature and severe pretension condition. In additions, irrespective of the thermal condition, curves for various modulus ratios tends to overlap each other, for both of the extreme pretension conditions ($k$=0，$\infty$). This apparently matches with the theoretical background of the presented linear solutions since both the modulus and thermal terms are missing for the cases of pure plate ($k$=0) and pure membrane ($k=\infty$) conditions.

### 3.3 Solutions of Lateral Slopes

The results of lateral slopes for both the room temperature ($T = 25^\circ\mathrm{C}$) and thermally-elevated condition ($T = 150^\circ\mathrm{C}$) are displayed in Figure 6 and 7 respectively. Those for the quasi-monolithic plate case ($B_1$=1.05) are found to agree completely with which of Sheplak and Dugundji [3] and thus provides a further check for the present developed approach. In additions, for the plate with high-modulus ratio ($B_1$=1.5), the corresponding edge zone is reasonably found to be slightly wider than which of nearly single-layered plate ($B_1$=1.05). On the other hand, the thermal effect on the lateral slope is obviously invisible.

## 4. CONCLUSION

The linear problem of large deflection of clamped and symmetrically layered isotropic plate due to lateral load is solved by the use of the modified Bessel functions. The solutions for the slightly heated quasi- monolithic plate ($B_1$=1.05, T=25) agree very well with those given by Sheplak and Dungundji [3]. Apparent effect of modulus' ratio can only be seen in a low pretension condition. The pretension effect becomes dominant, however, when a large pretension is considered rendering almost unified solutions for the structural responses regardless of the magnitude of modulus' ratio, unless it is an extremely high temperature condition. A stiffened plate tends to moderate the edge zone effect, with a similar definition for the edge region.

## 4. REFERENCES


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## 5. ACKNOWLEDGEMENT


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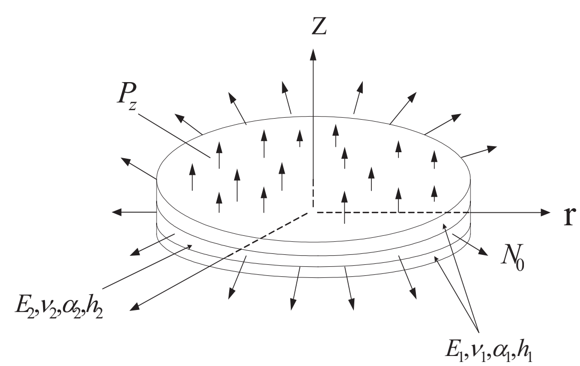
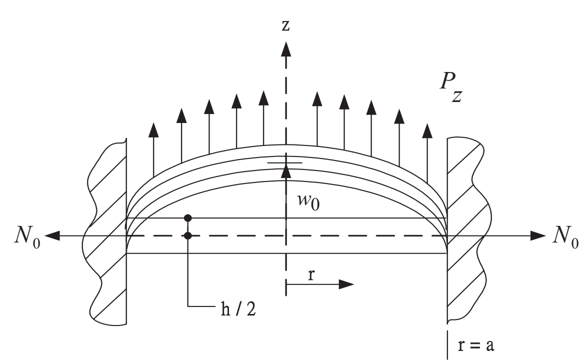



(a)




(b)

Fig. 1 Pre-stressed and Clamped Layered Plate in Large Deflection due to Uniform Lateral Load.

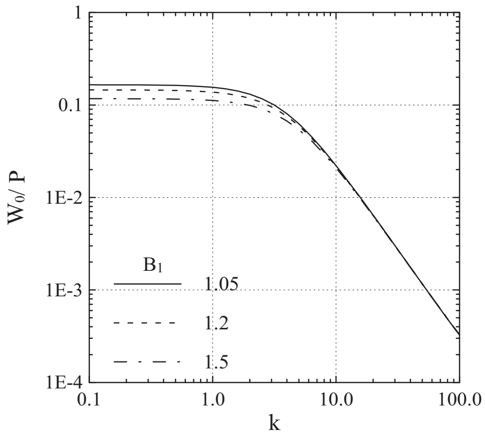



Fig. 2 Central Deflection vs. *k* for various Modulus Ratios, $T = 25^{\circ}\mathrm{C}$.

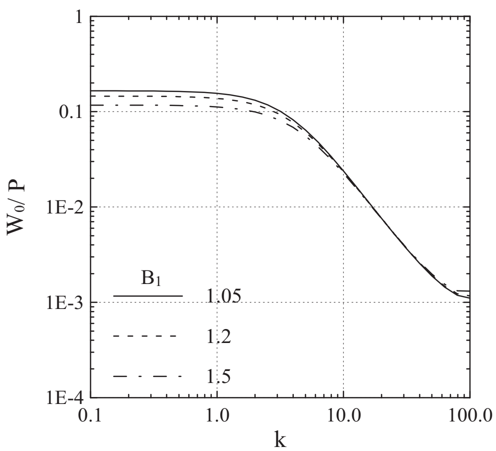



Fig. 3 Central Deflection vs. *k* for various Modulus Ratios, $T = 150^{\circ}\mathrm{C}$.

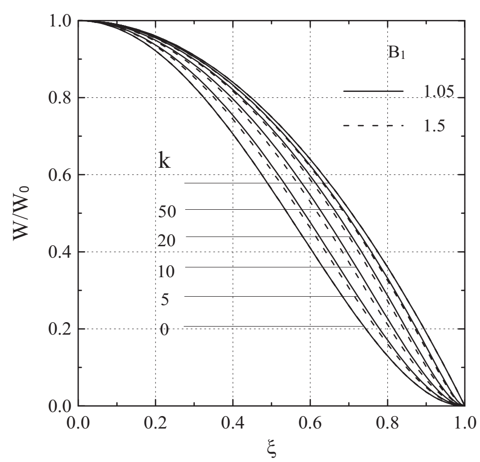



Fig 4. Global Deflection in Radial Direction, $T = 25^{\circ}\mathrm{C}$.

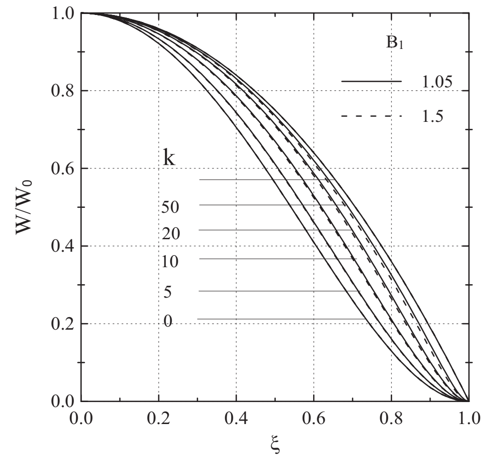



Fig 5. Global Deflection in Radial Direction, $T = 150^{\circ}\mathrm{C}$.

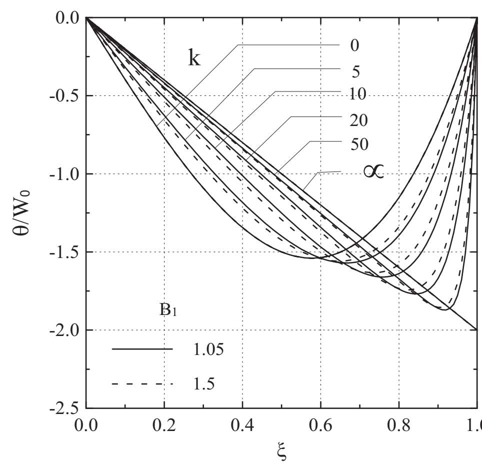



Fig 6. Global Slope in Radial Direction, $T = 25^{\circ}\mathrm{C}$.

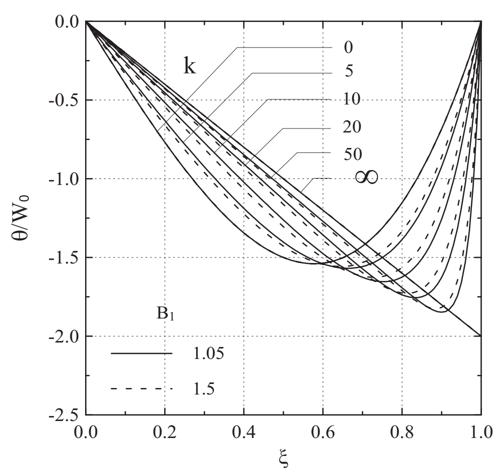



Fig 7. Global Slope in Radial Direction, $T = 150^{\circ}\mathrm{C}$