A method for adding static external forcing to invariant manifolds

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ABSTRACT

We present a new method for applying static forcing to invariant manifolds without breaking their invariance. This done by retaining only the applied force component tangential to the manifold surface. Outlined for a system with two degrees of freedom (dof) and exemplified through a six dof system, results show an improved performance compared to other methods of applying forcing directly on invariant manifolds. Means of estimating the error using the present method is presented. Typical areas of application are biasing or inclusion of dampening forces to reduced order models.

Keywords: invariant manifolds, reduced order model, external forcing

1 INTRODUCTION

Reduced order models based on invariant manifolds from nonlinear normal modes have been explored in many papers [1][2][3][4][5]. In this context, the term 'Invariant manifold' is used to classify all approximate solutions for modes not included directly in the reduced order model.
In [2], two possibilities for including external forcing is suggested: The first method includes external forces as new dofs and the second applies the external forces directly to the normal modes. The former results in a timedependent manifold, the latter destroys the manifold invariance. In our method, first outlined in [6], the forces are applied directly to the normal modes. However, to avoid breaking the manifold invariance, the applied force must be properly scaled and only the force component tangential to the manifold surface should be considered.

2 FORMULATION

To illustrate our concept, a two degrees of freedom system (n = 2) with quadratic and cubic nonlinearities in the displacement is considered. For simplicity, the system is decoupled to linear order. The equation of motion can then be written as [2]:

$$[\mathbf{I}]\ddot{\mathbf{u}} + [\boldsymbol{\Omega}]\mathbf{u} + [\alpha]\mathbf{u}^{2*} + [\beta]\mathbf{u}^{3*} = \mathbf{F} \tag{1}$$

where $\mathbf{u} = [u_1, u_2]^{\mathbf{T}}$ is a vector of modal degrees of freedom, [I] is a unit matrix, $[\boldsymbol{\Omega}]$ is a diagonal matrix of the squared ressonant frequencies ω_1 and ω_2, $\mathbf{u}^{2*}$ and $\mathbf{u}^{3*}$ are the second and third order Kronecker products of $\mathbf{u}$, respectively, α and β are matrices of nonlinear coefficients of size $n \times n^2$ and $n \times n^3$, respectively, and $\mathbf{F} = [F_1, F_2]$ is a vector of modal forces.
If we excite mode one, the retained mode, of our two-dof system, the modes can be approximated as follows:

$$X_1 = u_1 \tag{2}$$

$$\dot{X}_1 = v_1 \tag{3}$$

$$X_2 \approx f(u_1, v_1) \tag{4}$$

$$\dot{X}_2 \approx g(u_1, v1) \tag{5}$$

where X_i and $\dot{X}_i[i = 1, 2]$ are the approximate modal coordinates and velocities, respectively, accounting for the non-linear interactions. Note that X_2 and $Y_2 = \dot{X}_2$ are governed by u_1 and v_1 in this approximation. All possible solutions for the coordinate $u_2 \approx X_2 = f(u_1, v_1)$ can be projected to a surface in a subspace of the manifold space. An example of such a submanifold spanned by the modal coordinates u_1,v_1 and u_2 is illustrated in Figure 1. Using the approximate coordinates for u_2, v_2, the solution of Equation 1 with $\mathbf{F} = \mathbf{0}$ will be confined to the surface of the manifold. An example of this can be seen in Figure 1 where a sample simulation is illustrated along with the corresponding manifold. If $F_1 = F_2 \neq 0$, the solution using the approximate coordinates will still be confined to the manifold surface although the manifold no longer classify a valid set of solutions. Including the forces in the manifold has been explored in [2] and shown to result in a time varying manifold. This requires the forcing to be described by one or more harmonic functions and increases the size of the system to be simulated.
Using the method of scaling the force, a contribution from the non-retained mode must be included in the retained mode's forcing. Since, in the present approximation, the non-retained mode is required to reside in

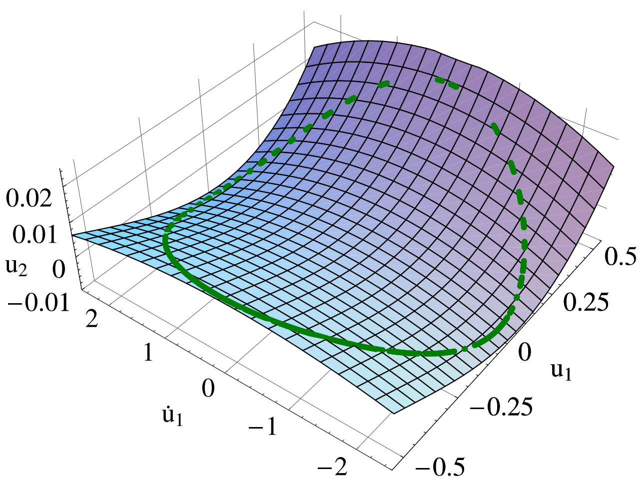

Figure 1: The projection of a 4D manifold surface into the 3D subspace spanned by $u_1, \dot{u}_1$ and u_2. As $u_2 \approx X_2 = f(u_1, v_1)$ where $v_1 = \dot{u}_1$, all approximate solutions for u_2 are confined to this surface. This is illustrated by the results from a simulation using the approximate coordinates for $u_2 \to X_2(\mathbf{u}_{\mathbf{S_M}}, \mathbf{v}_{\mathbf{S_M}})$.

the surface of the manifold, any force applied to this mode will also be required to work tangentially to the manifold surface. This suggests that the additional contribution to the retained mode from the non-retained mode should be equal to the force component tangential to the manifold surface. Since the modal forces are aligned along the phasespace axes u_1 and u_2, as illustrated in Figure 2, they can be projected onto the manifold and decomposed into the components tangential and perpendicular to the manifold surface. While the perpendicular component will try to 'deform' the manifold surface, thus breaking the manifold invariance, the tangential component will preserve its invariance. For weakly nonlinear systems, the tangential component will be dominant since the 'inclination' of the manifold will be negligible close to equilibrium ($u_1 = v_1 = 0$), and the error from disregarding the perpendicular component of the forcing can be assumed to be small.
To find the tangential vector, each coordinate is differentiated with respects to the system basis u_1 and u_2, parallell to the direction of the applied forces.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial X_1(u_1,v_1)}{\partial u_1} & \frac{\partial X_1(u_1,v_1)}{\partial u_2} \\ \frac{\partial X_2(u_1,v_1)}{\partial u_1} & \frac{\partial X_2(u_1,v_1)}{\partial u_2} \end{bmatrix} \quad (6)$$

As only mode one is included in the reduced system, the force will be scaled by the components in the first column of J. Thus the force to be applied is

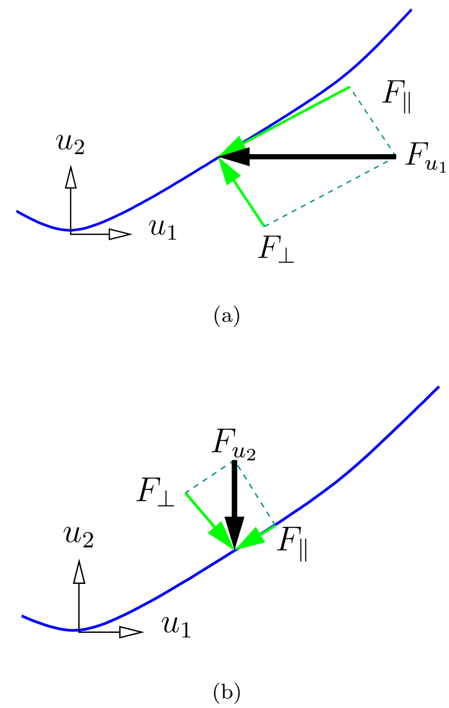

Figure 2: Decomposition of modal forces to components tangent ($F_\parallel$) and perpendicular ($F_\perp$) to the manifold surface. The resultant of the tangent forces is normalized, and the modal contributions scaled (Equation 7).

$$F_{tangent} = F_1 \frac{\partial X_1(u_1, v_1)}{\partial u_1} + F_2 \frac{\partial X_2(u_1, v_1)}{\partial u_1} \quad (7)$$

The resulting position and velocity dependent scaling factor is illustrated in Figure 3.

3 N DEGREES OF FREEDOM

Similar to finding one tangent vector in the above example with $n = 2$ the tangential vectors in a problem with $n > 2$ can be found from the Jacobian of a configuration space spanned only by the positions ($u_1 \cdots u_N$).

$$\mathbf{J} = \begin{bmatrix} \frac{\partial X_1(\mathbf{U}_{\mathbf{S_M}},\mathbf{V}_{\mathbf{S_M}})}{\partial u_1} & \cdots & \frac{\partial X_1(\mathbf{U}_{\mathbf{S_M}},\mathbf{V}_{\mathbf{S_M}})}{\partial u_n} \\ \frac{\partial X_2(\mathbf{U}_{\mathbf{S_M}},\mathbf{V}_{\mathbf{S_M}})}{\partial u_1} & \cdots & \frac{\partial X_2(\mathbf{U}_{\mathbf{S_M}},\mathbf{V}_{\mathbf{S_M}})}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_n(\mathbf{U}_{\mathbf{S_M}},\mathbf{V}_{\mathbf{S_M}})}{\partial u_1} & \cdots & \frac{\partial X_n(\mathbf{U}_{\mathbf{S_M}},\mathbf{V}_{\mathbf{S_M}})}{\partial u_n} \end{bmatrix} \quad (8)$$

where $X_i(\mathbf{U}_{S_M}, \mathbf{V}_{S_M})$ means that X_i is a function of all retained modes($\mathbf{S_M}$) . In the Jacobian, each column corresponds to a vector tangential to one submanifold. The tangential force to be applied to each of the

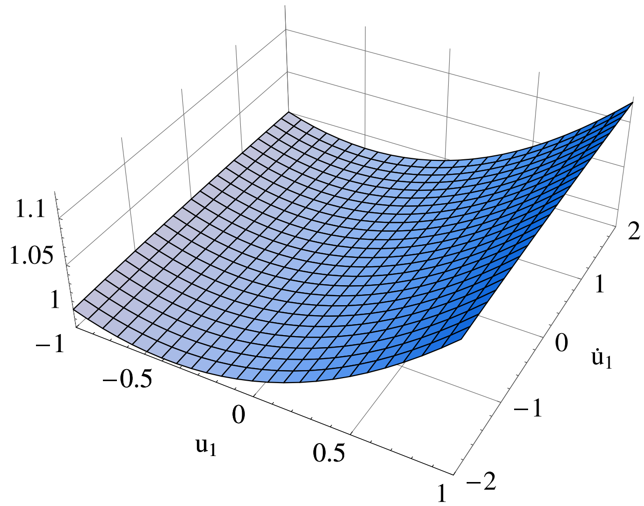

Figure 3: Position and velocity dependent scaling factor for the force applied to the retained mode.

retained modes is the projection of the resultant from all the forces onto the manifold. This means that for the first mode, the applied force is the product of the first column of the Jacobian (Equation 8) and the force corresponding to mode one, and so on for each mode. Genrealized, and only including the retained modes, this is

$$F_{scaled_j} = \sum_{i=1}^{n} F_i \frac{\partial X_i(\mathbf{U}_{S_M}, \mathbf{V}_{S_M})}{\partial u_j} \tag{9}$$

where j notes the $j'th$ mode, $j \in S_M$.

4 MATRIX FORMULATION

Considering the two dof system, the tangent and normal force components are:

$$\begin{bmatrix} F_\parallel \\ F_\perp \end{bmatrix} = \begin{bmatrix} \frac{\partial X_1(u_1,v_1)}{u_1} & \frac{\partial X_2(u_1,v_1)}{u_1} \\ \frac{\partial X_1(u_1,v_1)}{u_2} & \frac{\partial X_2(u_1,v_1)}{u_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{10}$$

which is

$$[\mathbf{F}_c] = [\mathbf{J}]^T [\mathbf{F}] \tag{11}$$

This is transformed back to the u_1 , u_2 basis to form the scaled force components

$$[\mathbf{F}]_{scaled} = [\mathbf{J}] [\mathbf{J}]^T [\mathbf{F}] \tag{12}$$

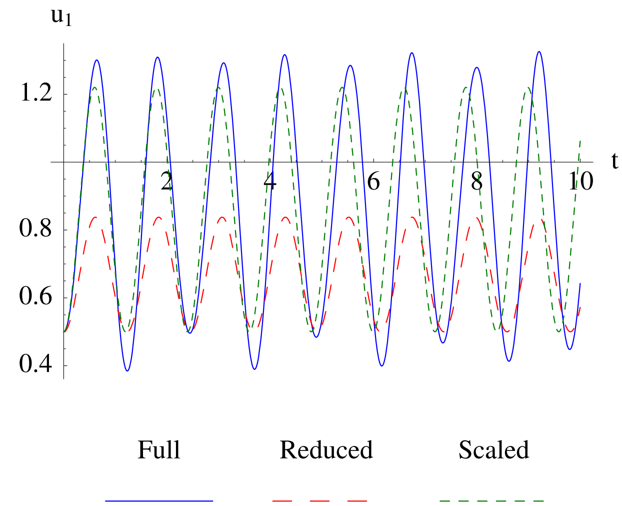

Figure 4: A plot of the amplitude vs. time for the retained mode from the sample two-mode system show how results from using the scaled forcing give results better than the simulations including forcing only to the retained mode.

which can be used in Equation 1 with the approximations for u_2 and v_2 and solved only for the retained modes.

5 RESULTS

Comparisons with full model simulations of the two-mode sample system show that using the method of scaled force provides better agreement than the results from the reduced order model including forcing only to the retained mode (Figure 4). A system of six dofs was built and simulated using the method presented in [2] using the computer algebra system Mathematica. The retained modes were included according to increasing resonant frequencies, starting with the lowest, and external forces were scaled sufficiently to excite the system without causing numeric instability for the full system. Results show (Figures 5 and 6) that the method using the scaled force clearly outperforms the method including only forces in the retained modes.

6 ERROR ESTIMATION

Disregarding the forces perpendicular to the manifold surface, some effects of the forcing are lost. This loss can be estimated by considering the effects of the perpendicular force components, whose scalar values indicate which of the excluded modes that gives the biggest loss and therefore will be the best candidate for inclusion

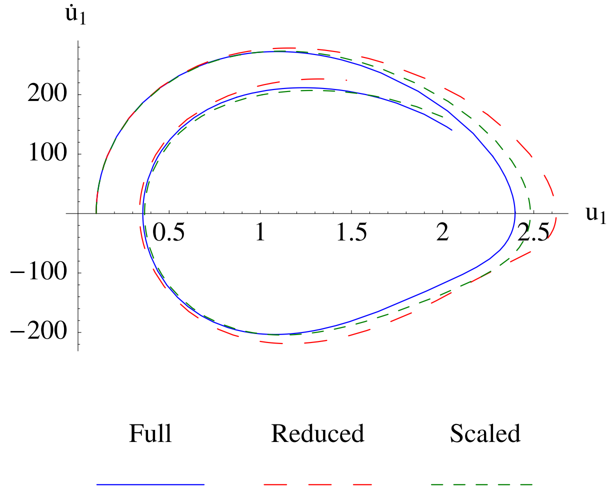

Figure 5: The phase plane of the first of the two retained modes from the 6 d.o.f system showing results for approximately the first one and a half cycle of the time-history results seen in Figure 6. The results from the model using the scaled forces clearly outperform the model where the forcing is taken to act on the retained modes only.

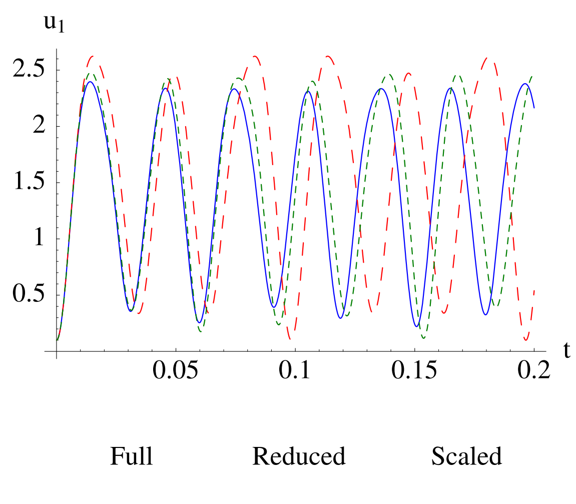

Figure 6: The amplitude vs time results for several cycles for the first of two retained modes from the 6 dof system. Again, the results from using the scaled forcing shows better agreement with the full model than does the model where the forcing is taken to act on the retained modes only.

in a reduced model with additional dofs. This method is still to be implemented.

7 CONCLUSION

A new and promising method for applying quasi-static forcing to invariant manifolds, without breaking their invariance, has been developed and is shown to give results that are better than those of the method presented in [2]. The only requirement for our method to be applied to a reduced order model is that it is differentiable with respect to the retained modal coordinates. The method should therefore be well suited for applying quasi-static forcing to semianalytic reduced order models developed from methods such as that presented in [2] . Our method also allows for an error estimation to be performed prior to running any simulations, by considering the magnitudes and the directions of the force components perpendicular to the surface of the manifold. Device examples and error estimation are currently being investigated.

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