Analysis of Temperature Effect on Diaphragm Resonant Sensors

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ABSTRACT

Resonant sensors and microsensors are one of the most common used sensors in industry. Temperature effect on the performance of such devices is of importance to the accuracy of measurement. In this paper, we are interested in the modeling of such diaphragm under thermal effect. We chose the dynamic analogue of Saint-Venant plate model under uniform temperature to analyze a circular diaphragm. The resulting equation turns out to be similar to the classic problem of plate stability under uniform in-plane loading. The modeling methodology can be applied to any type of thermal loading. It is found that under the assumption of isotropic materials with temperature-insensitive material properties, the frequency drift depends mainly on the temperature and the material properties are of no significant influence. Also, we show that the frequency-temperature relationship is nonlinear for the first mode but almost linear for higher modes.

Keywords: Circular plate, Microsensor, Thermal loading, Frequency drift.

1 INTRODUCTION

Resonant sensors are one of the most common used sensors in industry. The applications range from fluid, pressure to chemical sensing. The resonant sensors are composed mainly, of three major elements: the resonator, vibration drive and detection mechanisms. In the heart of the sensors is the resonator which may be a beam, bridge or a diaphragm [1]. The sensor is designed that the resonator's natural frequency is a function of the measurand. According to the application, the measurand alters the stiffness, shape, or mass of the resonator which will alter the resonator's resonant frequency. Temperature is one of the main environmental factors that influence the performance of such sensors. There is a wide interest in controlling, compensating such influence [2]. Investigation of the thermal frequency drift in resonant microsensors using beam elements was carried out in [3]. To the author's knowledge, no such attempt was done to the microsensors using plate diaphragm. The focus of this paper is on the analysis of such effect on a circular diaphragm resonant sensor. Modeling of the diaphragm varies in the literature, but mostly use of the Kirchoff linear plate model is quite common. In case of temperature field which is function of in-plane parameters and not of thickness, which the case for most thin diaphragms used in MEMS based sensors, such model does not work well. We resort to Saint-Venant thin plate model [4] for such application.

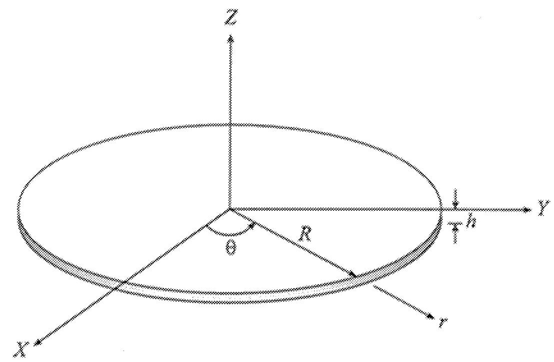

Figure 1: Plate geometry.

2 PROBLEM FORMULATION

We consider the coupled heat equation and the dynamic version of the Saint-Venant equations for a thermally excited circular plate. Figure 1 shows the geometrical characteristics of the plate. The thermal loading is assumed to be axisymmetric, and hence the plate vibrations are axisymmetric. They are governed by [5]

$$k\hat{\nabla}^2\hat{T} + \hat{Q} = \rho c_p \frac{\partial \hat{T}}{\partial \hat{t}} + \frac{E\alpha T_0}{1-2\nu}\frac{\partial \hat{e}}{\partial \hat{t}} \tag{1}$$

$$D\hat{\nabla}^4\hat{w} + \rho h \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} = \frac{1}{\hat{r}}\frac{\partial^2 \hat{w}}{\partial \hat{r}^2}\frac{\partial \hat{F}}{\partial \hat{r}} + \frac{1}{\hat{r}}\frac{\partial \hat{w}}{\partial \hat{r}}\frac{\partial^2 \hat{F}}{\partial \hat{r}^2} - \frac{1}{1-\nu}\hat{\nabla}^2\hat{M}_T \tag{2}$$

where

$$\hat{M}_T = E\alpha\int_{-\frac{1}{2}h}^{\frac{1}{2}h}\hat{T}(\hat{r},\hat{t})\hat{z}\,d\hat{z} = 0 \tag{3}$$

Here, $\hat{w}(\hat{r},\hat{t})$ is the plate transverse displacement, $\hat{F}(\hat{r},\hat{t})$ is the stress function, e is the dilatational strain due to the thermal effect, $\hat{T}(\hat{r},\hat{t})$ is the temperature distribution, ρ is the material density, h is the plate thickness, c_p is the heat capacity coefficient at constant pressure, E is the modulus of elasticity, α is the coefficient of thermal expansion, $\hat{Q}$ is the heat flux, and ν is Poisson's ratio. It is assumed here that the material properties are not functions of temperature.

For the compatibility relation, we note that

$$\epsilon_r = \frac{1}{Eh}(\hat{N}_r - \nu \hat{N}_\theta) + \alpha \hat{T} \tag{4}$$

$$\epsilon_\theta = \frac{1}{Eh}(\hat{N}_\theta - \nu \hat{N}_r) + \alpha \hat{T} \tag{5}$$

$$\hat{N}_r = \frac{1}{\hat{r}}\frac{\partial \hat{F}}{\partial \hat{r}} \text{ and } \hat{N}_\theta = \frac{\partial^2 \hat{F}}{\partial \hat{r}^2} \tag{6}$$

$$\epsilon_r = \frac{\partial \hat{u}}{\partial \hat{r}} \text{ and } \epsilon_\theta = \frac{\hat{u}}{\hat{r}} \tag{7}$$

where $\hat{u}$ is the radial displacement. It follows from Eqs. (4)-(7) that

$$\frac{\partial \hat{u}}{\partial \hat{r}} = \frac{1}{Eh}\left(\frac{1}{\hat{r}}\frac{\partial \hat{F}}{\partial \hat{r}} - \nu \frac{\partial^2 \hat{F}}{\partial \hat{r}^2}\right) + \alpha \hat{T} \tag{8}$$

$$\frac{\hat{u}}{\hat{r}} = \frac{1}{Eh}\left(\frac{\partial^2 \hat{F}}{\partial \hat{r}^2} - \frac{\nu}{\hat{r}}\frac{\partial \hat{F}}{\partial \hat{r}}\right) + \alpha \hat{T} \tag{9}$$

Eliminating $\hat{u}$ from Eqs. (8) and (9), we obtain the compatability equation

$$\hat{r}\frac{\partial^3 \hat{F}}{\partial \hat{r}^3} + \frac{\partial^2 \hat{F}}{\partial \hat{r}^2} - \frac{1}{\hat{r}}\frac{\partial \hat{F}}{\partial \hat{r}} = -Eh\alpha\hat{r}\frac{\partial \hat{T}}{\partial \hat{r}} \tag{10}$$

For boundary conditions, we consider a clamped plate. This case is important in most of MEMS devices, such as sensors and micropumps, as it is a more realistic representation of the actual boundary conditions. Therefore, the boundary conditions are

$$\hat{w} = 0 \text{ and } \frac{\partial \hat{w}}{\partial \hat{r}} = 0 \text{ at } \hat{r} = R \tag{11}$$

$$\hat{F} < \infty \text{ and } \hat{w} < \infty \text{ at } \hat{r} = 0 \tag{12}$$

$$\hat{u} = \frac{\partial^2 \hat{F}}{\partial \hat{r}^2} - \frac{\nu}{\hat{r}}\frac{\partial \hat{F}}{\partial \hat{r}} + E\alpha h \hat{T} = 0 \text{ at } \hat{r} = R \tag{13}$$

$$\hat{T} < \infty \text{ at } \hat{r} = 0 \;\; \hat{T} = T_0 \text{ at } \hat{r} = R \tag{14}$$

where R is the radius of the plate.

We consider the case in which the plate is exposed to zero-heat flux $\hat{Q} = 0$ and the temperature is kept constant at the plate edge at T_0.

We introduce nondimensional variables, defined as follows:

$$\hat{r} = Rr, \;\; \hat{t} = R^2\left(\frac{\rho h}{D}\right)^{1/2} t, \;\; \hat{T} = T_0 T,$$
$$\hat{w} = \frac{h^2}{R}w, \;\; \hat{z} = hz, \;\; \hat{F} = \frac{Eh^5}{R^2}F \tag{15}$$

Substituting Eq. (15) into Eqs. (1), (2), and (10)-(14), we obtain

$$\nabla^2 T = \Gamma_1 \frac{\partial T}{\partial t} + \Gamma_2 \frac{\partial e}{\partial t} \tag{16}$$

$$\frac{\partial^2 w}{\partial t^2} + \nabla^4 w = \epsilon\left[\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 F}{\partial r^2}\right] \tag{17}$$

$$r\frac{\partial^3 F}{\partial r^3} + \frac{\partial^2 F}{\partial r^2} - \frac{1}{r}\frac{\partial F}{\partial r} = -\frac{\alpha T_0 R^4}{h^4} r \frac{\partial T}{\partial r} \tag{18}$$

$$w = 0 \text{ and } \frac{\partial w}{\partial r} = 0 \text{ at } r = 1 \tag{19}$$

$$F < \infty \text{ and } w < \infty \text{ at } r = 0 \tag{20}$$

$$\frac{\partial^2 F}{\partial r^2} - \frac{\nu}{r}\frac{\partial F}{\partial r} + \frac{\alpha T_0 R^4}{h^4} T = 0 \text{ at } r = 1\,, \tag{21}$$

$$T < \infty \text{ at } r = 0 \text{ and } T = 1 \text{ at } r = 1 \tag{22}$$

where

$$\Gamma_1 = \frac{\rho c_p}{k}\left(\frac{D}{\rho h}\right)^{1/2}, \;\; \Gamma_2 = \frac{E\alpha}{(1-2\nu)k}\left(\frac{D}{\rho h}\right)^{1/2},$$
$$\text{and } \epsilon = \frac{12(1-\nu^2)h^2}{R^2}$$

The last two terms on the right-hand side of Eq. (15) represent the diffusion of heat and thermoelastic coupling.

Usually, the thermal diffusion and thermoelastic coupling terms are negligible because Γ_1 and Γ_2 are large enough to neglect the thermoelastic coupling term [6].

Hence, Eq. (16) is reduced to

$$\nabla^2 T = 0 \tag{23}$$

Solving Eq. (23) subject to the boundary conditions (22), we find that the nondimensional temperature distribution is given by

$$T(r,t) = 1 \tag{24}$$

Now, substituting Eq. (24) into Eqs. (17)-(21) yields

$$\frac{\partial^2 w}{\partial t^2} + \nabla^4 w = \epsilon\left[\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 F}{\partial r^2}\right] \tag{25}$$

$$r\frac{\partial^3 F}{\partial r^3} + \frac{\partial^2 F}{\partial r^2} - \frac{1}{r}\frac{\partial F}{\partial r} = 0 \tag{26}$$

$$w = 0 \text{ and } \frac{\partial w}{\partial r} = 0 \text{ at } r = 1 \tag{27}$$

$$F < \infty \text{ and } w < \infty \text{ at } r = 0 \tag{28}$$

$$\frac{\partial^2 F}{\partial r^2} - \frac{\nu}{r}\frac{\partial F}{\partial r} + \frac{\alpha T_0 R^4}{h^4} = 0 \text{ at } r = 1 \tag{29}$$

To this end, we solve the linear equation (26) subject to the boundary conditions (28) and (29). The general solution of the linear equation (26) can be expressed as

$$F = C_1(t)r^2 + C_2(t) \tag{30}$$

Using the boundary condition, Eq. (29), we find that $C_1(t) = -\frac{\alpha T_0 R^4}{2(1-\nu)h^4}$ and $C_2(t)$ is an arbitrary function of time.

Next, we let

$$F = \Phi - \frac{\alpha T_0 R^4}{2(1-\nu)h^4}r^2 + C_2(t) \tag{31}$$

into Eqs. (25)-(29) and obtain

$$\frac{\partial^2 w}{\partial t^2} + p\nabla^2 w + \nabla^4 w = \epsilon\left[\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial \Phi}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 \Phi}{\partial r^2} - 2c\frac{\partial w}{\partial t}\right] \tag{32}$$

$$r\frac{\partial^3 \Phi}{\partial r^3} + \frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r}\frac{\partial \Phi}{\partial r} = 0 \tag{33}$$

$$w = 0 \text{ and } \frac{\partial w}{\partial r} = 0 \text{ at } r = 1 \tag{34}$$

$$w < \infty \text{ and } \Phi < \infty \text{ at } r = 0 \tag{35}$$

$$\frac{\partial^2 \Phi}{\partial r^2} - \frac{\nu}{r}\frac{\partial \Phi}{\partial r} = 0 \text{ at } r = 1 \tag{36}$$

where

$$p = \frac{12\alpha T_0(1+\nu)R^2}{h^2}.$$

3 EIGENVALUE PROBLEM

Next, we assume a harmonic response of the form

$$w(r,t) = \phi(r)e^{i\omega t} \tag{37}$$

and obtain the following eigenvalue problem for the nondimensional mode shapes $\phi_j(r)$ and corresponding nondimensional natural frequencies ω_j:

$$\nabla^4\phi + p\nabla^2\phi - \omega^2\phi = 0 \tag{38}$$

$$\phi = 0 \quad \text{and} \quad \frac{d\phi}{dr} = 0 \quad \text{at} \quad r = 1 \text{ and } \phi < \infty \text{ at } r = \tag{39}$$

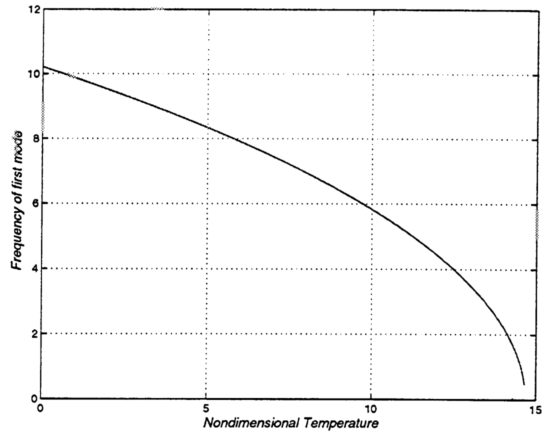

Figure 2: Variation of the first nondimensional frequency with the nondimensional parameter (temperature) p.

Therefore, Eq. (32) becomes

$$\nabla^4\phi + p\nabla^2\phi - \omega^2\phi = 0 \tag{40}$$

The general solution of Eq. (40) can be expressed in terms of Bessel functions as

$$\phi(r) = A_1 J_0(\xi_1 r) + A_2 Y_0(\xi_1 r) + A_3 I_0(\xi_2 r) + A_4 K_0(\xi_2 r) \tag{41}$$

where

$$\xi_1 = \sqrt{\tfrac{1}{2}\left[p + \sqrt{p^2 + 4\omega^2}\right]} \tag{42}$$

$$\xi_2 = \sqrt{\tfrac{1}{2}\left[-p + \sqrt{p^2 + 4\omega^2}\right]} \tag{43}$$

Substituting Eq. (41) into the boundary conditions, yields that $A_2 = A_4 = 0$. which can be solved numerically for the ω_n as a function of p and hence T_0. In Figures 2-4, we show variation of the first three nondimensional frequencies with p. Buckling occurs when the first natural frequency is zero, which corresponds to $p = -14.68$. It is clear from Figure 2 that the variation of the fundamental (first) frequency with temperature is nonlinear. In Figures 3 and 4, the variation is almost linear which indicates that for higher frequencies, the frequency variation with temperaure is almost linear.

4 CONCLUSIONS

We presented a methodology for deriving the equation that model a microsensor isotropic diaphragm under thermal loading. The method is applicable to any type of thermal loading. We chose the uniform temperature field as a case study in this paper. We solved the eigenvalue problem to obtain the relation between the frequency and the temperature. It is found that under the assumption of an isotropic diaphragm, the frequency change depends only on the temperature and does not depend on the material properties. Also, it is found that the relation between the fundamental frequency and the temperature is of nonlinear type, but for the higher frequencies the relation is more of a linear type. The above frequency-temperature diagrams are of value to the designers because of the nondimensional character of the temperature. Another important result is the occurrence of buckling at a certain temperature, which should be avoided in the working environment of the microsensor.

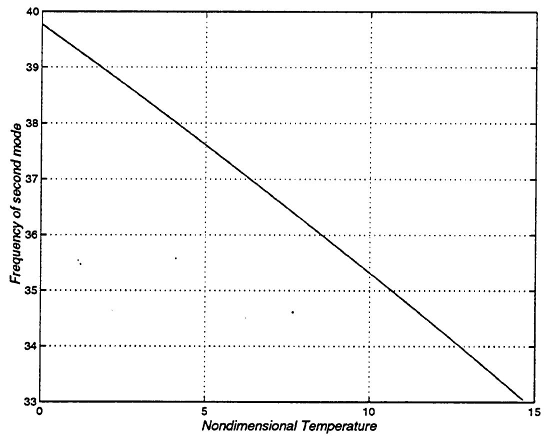

Figure 3: Variation of the second nondimensional frequency with the nondimensional parameter (temperature) p.

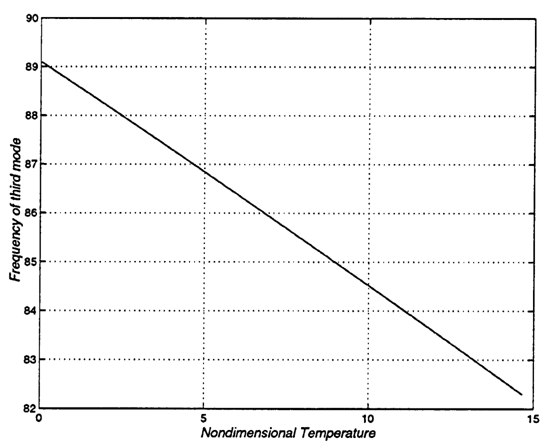

Figure 4: Variation of the third nondimensional frequency with the nondimensional parameter (temperature) p.

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