Analytical Modeling of beam behavior under different actuations

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ABSTRACT

The aim of this paper is to establish simple analytical equations of the beam deflection for different actuation methods (magnetic with current, magnetic with magnet, electrostatic and piezoelectric). The analytical formulae are obtained from the study of the physical phenomena. Although the results presented are obtained by making certain approximations, this study provides order of magnitude results for the deflection, and most importantly quantifies the influence of each parameter (current, voltage, thickness, length, etc.).

Keywords: Modeling, beam, electrostatic, magnetic, piezoelectric

STRUCTURE AND ACTUATION

We consider a beam which is free at one end and built-in at the other. The width is denoted by b, the length L and the thickness h. The four actuation methods are presented in the following sections.

Magnetic actuation

Two magnetic actuations are studied:
- Generation of a force located at the beam extremity by placing the system in an area where there is a magnetic field and causing a current to flow in a conductor deposited on the beam (Fig. 1). This force can be controlled by varying of both the current I and the magnetic field B.

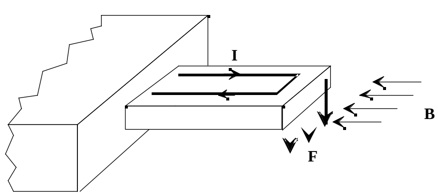

Figure 1: Magnetic actuation (current)

- Generation of a couple on the beam surface by placing it in a magnetic field and by depositing a magnetized layer on the surface (Fig. 2). This can only be controlled by varying of the magnetic field B. The generated couple is not uniform and depends on the deflection of the considered point. Two cases arise according to the technology used to implement the magnetized layer. We will see that these two cases lead to the same theoretical approach.

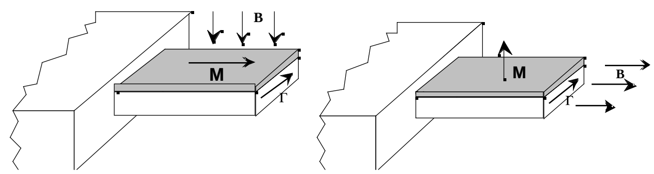

Figure 2: Magnetic actuation (magnet)

Electrostatic actuation

The electrostatic actuation of the beam can be achieved by applying a potential difference between the beam and a rigid electrode placed in front of the beam (Fig. 3). In this case, the electrostatic pressure is applied to the entire beam surface and is not uniform (it depends on the deflection of the considered point).

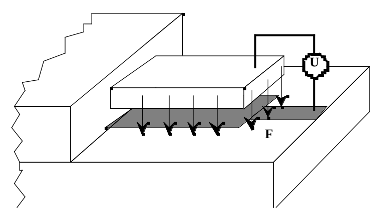

Figure 3: Electrostatic actuation

Piezoelectric actuation

The piezoelectric actuation of the beam can be done by applying a potential difference to a piezoelectric material deposited on the beam (Fig. 4). The elongation of the piezoelectric layer causes the beam deflection.

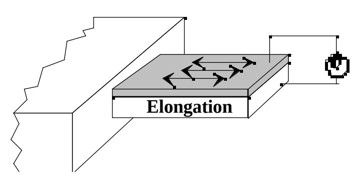

Figure 4: Piezoelectric actuation

ANALYTICAL PHYSICAL EQUATIONS

The differential equation governing the deflection of the beam is [1]:

$$\frac{d^2w/dx^2}{\left(1+\left(dw/dx\right)^2\right)^{3/2}} = -\frac{M}{E\,I_z} \tag{1}$$

Where w is the deflection for each beam point of abscissa x ($x \in [0, L]$), M the bending moment in the section of abscissa x, I_z the inertia moment of the section of the neutral axis z and E the modified Young modulus (Appendix III).

With the assumption of small deflections $\frac{dw}{dx} << 1$, the equation (1) becomes: $\frac{d^2w}{dx^2} = -\frac{M}{E\,I_z}$

Magnetic actuation (current)

With this type of actuation, the beam is subject to a single force which is located at the beam extremity. The result then comes from classical mechanics (Appendix I) with a force magnitude F:

$$F = NIbB \tag{2}$$

where N is the number of turns, I the current intensity, B the magnetic field and b the beam width.

$$w(x) = \frac{6\,N\,I\,B}{E\,h^3}x^2\left(L - x/3\right) \tag{3}$$

Note: This equation assumes that:

$$\frac{6\,NI\,B}{Eh^3}L^2 << 1 \tag{4}$$

Magnetic case (magnet)

With this type of actuation, each section of magnetic material (length dx) has one dipole moment $d\vec{m}$. Its modulus is:

$$dm = \mathrm{M}\,h_2\,b\,dx \tag{5}$$

where M is the magnetization and h_2 the magnetic material thickness.

The magnetic field B thus creates a couple given as follows:

$$d\vec{\Gamma} = d\vec{m} \times \vec{B} \tag{6}$$

For small deflections, the angle between the dipole moment and the magnetic field can be considered constant and equal to $\pi/2$. Then, the bending moment in a section of x-abscissa is:

$$M = -\int_x^L d\Gamma = -\int_x^L \mathrm{M}h_2bB\,dx = -\mathrm{M}h_2bB(L - x) \tag{7}$$

From appendix I, we can see that this type of actuation is equivalent to the actuation with a single force F at the beam extremity:

$$F = \mathrm{M}h_2\,bB \tag{8}$$

Appendix I therefore applies, but the force F is applied not only to the beam but to the combined form of the beam and the magnet. We can thus apply the results of appendix I on the condition that we consider the geometrical properties of a homogeneous beam to be equivalent to the combined form. This equivalent beam will consist of only one material (the same one as the substrate: modified Young modulus E_1) and will not have a rectangular section but a T-shaped section: the lower portion will have a width $b_1 = b$ and a thickness h_1 (thickness of the real beam) and the higher portion will have a width $b_2 = bE_2/E_1$ (where E_2 is the modified Young modulus of the magnetic material) and a thickness h_2. The expression of the inertia moment of this equivalent beam is then:

$$I_{eq} = \frac{b}{12E_1}\frac{h_1^4E_1^2 + h_2^4E_2^2 + 2h_1h_2E_1E_2\left(2h_1^2 + 3h_1h_2 + 2h_2^2\right)}{h_1E_1 + h_2E_2} \tag{9}$$

Using appendix I with the expressions (8) and (9), we obtain the deflection expression:

$$w(x) = \frac{6\,\mathrm{M}Bh_2\left(E_1h_1 + E_2h_2\right)\left(Lx^2 - x^3/3\right)}{h_1^4E_1^2 + h_2^4E_2^2 + 2h_1h_2E_1E_2\left(2h_1^2 + 3h_1h_2 + 2h_2^2\right)} \tag{10}$$

Note : This expression is only valid if:

$$\frac{6\,\mathrm{M}Bh_2\left(E_1h_1 + E_2h_2\right)L^2}{h_1^4E_1^2 + h_2^4E_2^2 + 2h_1h_2E_1E_2\left(2h_1^2 + 3h_1h_2 + 2h_2^2\right)} << 1 \tag{11}$$

Electrostatic case

With this type of actuation, the beam is subject to a non-uniform pressure on all the surface facing the electrode. The electrostatic pressure is not uniform because it depends on the deflection point of the beam considered:

$$P = P_o \Big/ \left(1 - w/e\right)^2 \tag{12}$$

with:

- $e = d_1 + d_2/\varepsilon_2$ air-gap equivalent to the space between the beam and the electrode (d_1 real gap between the beam and the insulator, d_2 insulator thickness and ε_2 relative dielectric constant of the insulator material).
- $P_o = \dfrac{\varepsilon_0 U^2}{2e^2}$ electrostatic pressure when the beam is not deformed (U is the applied voltage and ε_0 is the dielectric constant).

In this case, the bending moment is:

$$M = -\int_x^L \frac{P_o\, b(X-x)}{\left(1 - w(X)/e\right)^2} dX \tag{13}$$

And the differential equation to solve to find the beam profile is thus:

$$\frac{d^2 w}{dx^2} = -\frac{M}{E I_z} = \frac{12 P_o}{E h^3} \int_x^L \frac{(X-x)}{\left(1 - w(X)/e\right)^2} dX \tag{14}$$

There is no obvious analytical solution to this equation. Thus, we propose using an iterative method to find an approximation to the analytical solution.

When a voltage U is applied between the beam and the electrode, the beam is not deformed: the electrostatic pressure can thus initially be considered constant and equal to P_o. According to the results presented in appendix II, the beam will become deformed under the action of this constant pressure and the equation of the beam profile is: $w_o = P_o / k(x)$ with $k(x) = 2Eh^3 \Big/ \left(x^4 - 4Lx^3 + 6L^2x^2\right)$. The electrostatic pressure is then no longer constant: its expression depend on the x-abscissa $P_1 = P_o \Big/ \left(1 - w_o/e\right)^2$. The deflection due to this unconstant pressure is approached by: $w_1 = P_1 / k(x)$, etc.

Then $w_{n+1} = \dfrac{P_{n+1}}{k(x)} = \dfrac{P_o}{k(x)\left(1 - w_n/e\right)^2} = \dfrac{w_o}{\left(1 - w_n/e\right)^2}$. If the solution is convergent, w is the solution of:

$$w^3 - 2ew^2 + e^2 w - e^2 w_o = 0 \tag{15}$$

The analytical solution of this polynomial equation is:
if $w_o(x) < 4e/27$ then

$$w(x) = \frac{2e}{3}\left[1 - \cos\left(\frac{1}{3} Arcos\left(1 - \frac{27 w_o(x)}{2e}\right)\right)\right] \tag{16}$$

if $w_o(x)$? $4e/27$ then

$$w(x) = d_1 \tag{17}$$

The pull-in voltage U_{p-i} is obtained when:

$w_o(L) = 4e/27$, i.e. $U_{p-i} = \dfrac{4}{9L^2}\sqrt{\dfrac{Eh^3 e^3}{\varepsilon_0}}$

The pull-out voltage U_{p-o} is obtained when:

$k(L) d_1 > \dfrac{\varepsilon_0 \varepsilon_2^2 U^2}{2d_2^2}$, i.e. $U_{p\text{-}o} = \dfrac{2d_2}{\varepsilon_2 L^2}\sqrt{\dfrac{Ed_1 h^3}{3\varepsilon_0}}$

Note : This expression is only valid if:

$$\frac{-\dfrac{9P_oL^3}{Eh^3}\sin\left(\dfrac{\pi}{3} - \dfrac{1}{3}a\cos\left(-1 + \dfrac{81P_oL^4}{4eEh^3}\right)\right)}{\sqrt{-1 - \left(-1 + \dfrac{81P_oL^4}{4eEh^3}\right)^2}} << 1 \tag{18}$$

Piezoelectric case

Let us consider an element of the system taken between two sections x and $x + dx$. The internal forces of the piezoelectric material section are reduced to a force F_2 and a bending moment M_2. Similarly, the internal forces of the beam are reduced to a force F_1 and a bending moment M_1. By balancing forces and moments on a section, we obtain the two following equations:

$$F_1 + F_2 = 0 \quad \text{and} \quad F_1\left(h_1 + h_2\right)/2 + M_1 + M_2 = 0 \tag{19}$$

where $M_1 = E_1 I_1 / r$ and $M_2 = E_2 I_2 / r$, r is the curvature radius, E_1 and E_2 are the Young's moduli of the beam material and of the piezoelectric material, I_1 and I_2 are the inertia moment of the beam and of the piezoelectric layer, h_1 and h_2 are respectively the thickness of the beam and the thickness of the piezoelectric layer. Combining equations 19, we obtain:

$$F_1 = -2\frac{E_1 I_1 + E_2 I_2}{r\left(h_1 + h_2\right)} \tag{20}$$

The relative lengthening of piezoelectric material and beam material must be the same at their interface:

$$d_{31}\frac{U}{h_2}+\frac{1}{E_2}\frac{F_2}{h_2 b}+\frac{h_2}{2r}=\frac{1}{E_1}\frac{F_1}{h_1 b}-\frac{h_1}{2r} \quad (21)$$

Where U is the applied voltage and d_{31} the modified piezolectric constant (Appendix III).

Combining equations 19- 21, we obtain:

$$-\frac{1}{r}=\frac{d_{31}U}{h_2}\frac{2}{h_1+h_2+4\frac{(E_1I_1+E_2I_2)(E_1h_1+E_2h_2)}{bh_1h_2E_1E_2(h_1+h_2)}} \quad \frac{d^2w}{dx^2}$$

(22)

The solution of this equation gives the deflection profile:

$$w(x)=\frac{3\, d_{31}\, U\, E_1E_2\, h_1\,(h_1+h_2)\,x^2}{h_1^4E_1^2+h_2^4E_2^2+2h_1h_2E_1E_2\,(2h_1^2+3h_1h_2+2h_2^2)} \quad (23)$$

This equation is similar to the equation presented in [2].

We have seen that the radius of curvature is constant. The piezoelectric case is then equivalent to the actuation of an equivalent T-shaped beam with an inertia moment I_{eq} (eq. 9) with a constant bending moment M_{eq} :

$$M_{eq}=-\frac{d_{31}\,U\,bE_1E_2\,h_1(h_1+h_2)}{2(h_1E_1+h_2E_2)} \quad (24)$$

Note: As for the other actuations, the deflection expression can be used only if:

$$-\frac{6\, d_{31}\, U\, E_1E_2\, h_1\,(h_1+h_2)\,L}{h_1^4E_1^2+h_2^4E_2^2+2h_1h_2E_1E_2\,(2h_1^2+3h_1h_2+2h_2^2)}<<1 \quad (25)$$

PARAMETER ANALYSIS

Taking into account eq. 3 we can see that the beam deflection in the magnet actuation (current) is proportional to the magnetic field B, to the current I, to the number of turns N and is inversely proportional to the cube beam thickness. Moreover, it is independent of the beam width b.

In view of eq. 10 we know that with a magnetic actuation with magnet, the beam deflection is proportional to the magnetic field B and to the magnetization M and independent of the beam width b. The effects of layer thickness (h_1,h_2), beam length L and the modified Young moduli (E_1,E_2) are not clear from this expression. The maximal beam deflection $(x = L)$ is proportional to L^3 .

In the electrostatic actuation (eq. 16), the parameter dependency is not straightforward. But it should be noted that the voltage U, the length L and the thickness h only appear associated in the term $(L^4U^2)/(Eh^3)$.

With the piezoelectric actuation (eq. 23), the beam deflection is proportional to the voltage U, to the modified piezoelectric coefficient d_{31}. And the maximal beam deflection $(x = L)$ is proportional to L^2 .

CONCLUSION

As presented in the objective, these four analytical expressions for the beam deflection can be used to find the effect of each parameter and to obtain an order of magnitude for the beam deflection.

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APPENDIX

I. *Beam deflection with a punctual force at the extremity*

$M=-F(L-x)$, then $w=\frac{F}{2E\,I_z}x^2(L-x/3)$

II. *Beam deflection with a uniform pressure P*

$M=-\int_x^L Pb\,(X-x)dX=-\frac{Pb}{2}(L-x)^2$ then

$$w=\frac{Pb}{24\,E\,I_z}\left(x^4-4\,Lx^3+6\,L^2x^2\right)$$

III. *Modified parameters*

since $h<<b$ and $h<<L$, the modified Young modulus and the modified piezoelectric constant are:

$E_{mod}=E/(1-\nu^2)$ *and* $d_{mod}=(1+\nu)d_{31}$