Large-Deflection Beam Model for Schematic-Based Behavioral Simulation in NODAS

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ABSTRACT

MEMS design in NODAS is based on a geometrically intuitive schematic representation of MEMS using a small set of atomic elements (anchors, beams, plates and electrostatic gaps), each of which is associated with a geometrically parameterized lumped behavioral model. A linear beam model is unsuitable for simulation of devices where the beam deflection is large or where axial stress exists in the beam. Due to the complexity of the nonlinear beam mechanics, exact analytical solutions are generally unavailable. A nonlinear beam model for behavioral simulation is presented, which captures the geometric nonlinearity caused by large axial stress and large deflection. Very good agreement of the NODAS simulation results of example problems with available analytical solutions and with FEA simulation results verifies the model accuracy.

Keywords: beam, geometric nonlinearity, axial stress, large deflection.

1 INTRODUCTION

NODAS (NOdal Design of Actuators and Sensors) is a schematic-based MEMS design methodology being developed at Carnegie Mellon University [1]. The methodology is based on a geometrically intuitive hierarchical representation of MEMS using a small set of atomic elements (anchors, beams, plates and electrostatic gaps), each of which is associated with a geometrically parameterized lumped behavioral model written in an analog hardware description language. Implementation within the Cadence Design Environment integrates behavioral models written in VerilogA with electronics/MEMS schematic capture, fast behavioral DC, AC, and transient simulations, DRC, layout generation, and layout extraction [2]. Similar methodologies include UCB SUGAR in MATLAB [3] and Coventor's ARCHITECT in Saber [4].

The beam is a key element of the NODAS library. The linear beam is modeled with a linear stiffness matrix K, where the axial displacements are independent of lateral deflection. These assumptions are only acceptable in applications where the displacements are small. Although the linear model covers the basic beam bending mechanics in a large number of surface-micromachined devices such as folded-flexure resonators and accelerometers, it is unsuitable for simulation of devices such as large stroke actuators where the beam nonlinearity is non-negligible. Beam nonlinearity arises from two main effects: material nonlinearity, where the relation between the strain and the stress is nonlinear; and geometric nonlinearity, where the material is linear elastic but the relation between force and displacement of an element is nonlinear. The discussion in this paper is restricted to geometric nonlinearity.

Analytical solutions are available for several special cases of geometric nonlinearity such as a cantilever beam under single concentrated lateral load at the free end (the exact shape of the elastic curve is called the "elastica") [5], a cantilever beam under both lateral force and moment [6], and a fixed-fixed beam with a single concentrated lateral load at the midpoint of the beam [7]. Due to the complexity of the nonlinear beam mechanics and the inability to exploit superposition in nonlinear problems, analytical solutions are only available for these special simple cases. Nonlinear beams can be analyzed using finite element methods, however these methods do not fully support hierarchical mixed-domain simulations, especially for fast transient analysis and analysis with electronics. Other prior work on nonlinear beam behavioral modeling has concentrated on stress-stiffening effects for small angle deflection [8].

In this paper, we present a lumped parameterized nonlinear beam behavioral model for use within the conventional analog behavioral simulators. The model is reusable and composable hence can be used as the building block for complicated systems. It handles both large axial stress and large deflection.

2 NONLINEAR BEAM MODEL

Typical applications involving geometric beam nonlinearity are large stroke actuators, in which the beams are stiffened due to large deflection, and fixed-fixed beam devices, in which the beam starts to behave nonlinearly at very small displacements due to large axial stresses [9]. These cases represent the main sources of geometric nonlinearity: large axial stress and large geometric deflection.

2.1 Large axial stress

The mechanics of nonlinear beam bending with axial stress has been studied extensively [6] and is given by the Euler-Bernoulli equation:

$$\frac{d^4 y}{dx^4} - N\frac{d^2 y}{dx^2} = 0, \tag{1}$$

where the axial force, N, is the external force applied to the beam in its axial direction and only concentrated loads are applied. A geometric stiffness matrix K_G is then derived

using the energy methods [10], yielding a nonlinear force-displacement relation:

$$\begin{bmatrix} F_{xa} \\ F_{ya} \\ M_{za} \\ F_{xb} \\ F_{yb} \\ M_{zb} \end{bmatrix} = \left(\underbrace{\begin{bmatrix} \frac{EA}{L} & & & & & \\ 0 & \frac{12EI}{L^3} & \textit{symmetric} & & & \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & & & \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & & \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{K_0} + N \underbrace{\overbrace{\begin{bmatrix} 0 & \textit{symmetric} & & & & \\ 0 & \frac{6}{5L} & & & & \\ 0 & \frac{1}{10} & \frac{2L}{15} & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -\frac{6}{5L} & -\frac{1}{10} & 0 & \frac{6}{5L} & \\ 0 & \frac{1}{10} & -\frac{L}{30} & 0 & -\frac{1}{10} & \frac{2L}{15} \end{bmatrix}}^{K_G}}_{K_{G0}} \right) \begin{bmatrix} x_a \\ y_a \\ \phi_a \\ x_b \\ y_b \\ \phi_b \end{bmatrix}, \quad (2)$$

where a and b subscripts represent the two ends of the beam, K_0 is the linear stiffness matrix, and the axial force is

$$N = \frac{EA}{L}(x_b - x_a) . \quad (3)$$

Consider a fixed-fixed beam example with a non-zero axial force; K_G is then a non-zero matrix, and the beam is stiffened. If we apply (3) directly to a fixed-fixed beam with a force F_y in the middle (Figure 1), we will see that due to the symmetric boundary condition at the center of the beam, x displacements are forced to be zero, leading to a null axial force N and hence null K_G, which is not true physically. Equation (3) has to be modified in order to correctly model the internal axial stress.

When a beam is in tension, the actual beam length L' is longer than the original length L. Although there is no displacement in the x direction at the beam ends, the bending from tensile stress generates an axial force

$$N = \frac{EA}{L}(L' - L) , \quad (4)$$

where L' is the actual length along the center line of the beam. This length is calculated by integrating the arc length ds along the curved beam based on the cubic shape functions for small angle beam bending, $y(x)$:

$$L' = \int ds = \int_{x_a}^{L + x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx . \quad (5)$$

The axial force can be substituted into (2) as the axial force in x-direction only when the lateral deflection is small, *i.e.*, it is only valid for small angle deflection.

Now consider a cantilever beam loaded with a lateral force F_y at the free end and no external axial force F_x. In (2), the axial elements in K_G (row 1 and 4) are all zeros, resulting in zero x displacement, as in the linear case shown in

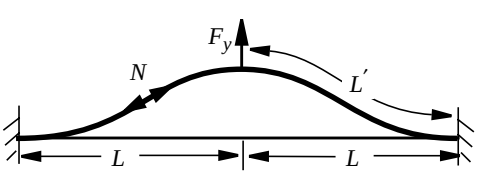

Figure 1: Lateral deflection of a fixed-fixed beam.

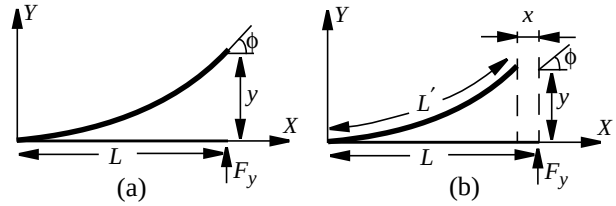

Figure 2: Cantilever beam (a) linear case (b) actual case (elastica).

Figure 2(a). However, the cantilever beam is free to bend in the x direction, thus the axial stress in the cantilever beam is zero for small deflection and the beam length remains unchanged. The self-consistent solution (Figure 2(b)), is that the nonlinear displacement in x comes from geometric foreshortening of the beam, where $L' = L$. To include this effect, the force-balance equation for the axial direction in (2) must be altered to $F_{xa} = -F_{xb} = N$, where N is calculated from (4).

2.2 Large deflection

The across and through variables at the connection terminals are the variables used to form the system matrix [1]. They represent the displacements and forces/moments of the elements in the chip frame. Communication between the elements is thus performed in the chip frame, following the Kirchhoff's network laws specified by the topology.

To deal with large deflection accurately, a physical beam should be composed of multiple beam elements. As is the case with finite elements, this allows the element cubic shape functions to more closely fit the actual shape function of the structure. These elements are related through dynamic coordinate transformations, as illustrated in Figure 3.

The beam mechanics stated in previous sections is only valid in the local frame of each beam element, therefore, the displacements and forces/moments in the chip frame must first be transformed into the local frame before being applied to the force-balance equation, then be transformed back to the chip frame to join the analysis of the entire network. The transformation is done through the coordinate translation about the left end of the beam, node a, and the rotational coordinate transformation about the angular displacement of the left end in the chip frame, ϕ_a:

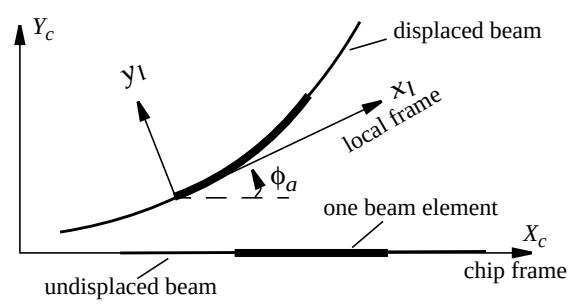

Figure 3: Dynamic coordinate transformation.

$$\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_l = \underbrace{\begin{bmatrix} \cos\phi_a & \sin\phi_a & 0 \\ -\sin\phi_a & \cos\phi_a & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T} \left(\begin{bmatrix} x \\ y \\ \phi \end{bmatrix}_c - \begin{bmatrix} x_a \\ y_a \\ \phi_a \end{bmatrix}_c \right). \quad (6)$$

As long as enough beam elements are used to compose the physical beam, the displacements of the beam ends in the local frames of each beam segment will be small enough to satisfy the small deflection assumption of the nonlinear beam model. Since ϕ_a is a variable varying in time along with the beam bending dynamics, the coordinate transformation stated in (6) is a dynamic transformation rather than the static coordinate transformation used in the linear beam model [1].

The beam length integral (5) is also calculated in the local frame, because small angle deflection is the assumption used in the derivation of the cubic shape functions. The explicit solution to the integral is implemented using a first-order Taylor series expansion. Both the displacements used in the shape functions and the limits of the integration must be the values measured in the local frame, as shown in Figure 4.

2.3 Numerical implementation

Because of the small angle deflection assumption, an incremental load method is needed to obtain accurate simulation results for large deflections. A nonlinear numeric method, such as Newton-Raphson, is also needed to solve the nonlinear equations. These techniques have to be included in the numeric solver, as done in commercial FEA tools and in behavioral simulation tools with self-maintained numeric solvers such as SUGAR. Since the NODAS models are embedded in Cadence, we take advantage of the incremental loading method and Newton-Raphson already included in the Spectre simulator. Accurate nonlinear solution can be obtained either by sweeping the load or by setting initial values at critical nodes.

3 SIMULATION VERIFICATION

To verify the nonlinear beam model, multiple spring topologies, including the cantilever beam, the fixed-fixed

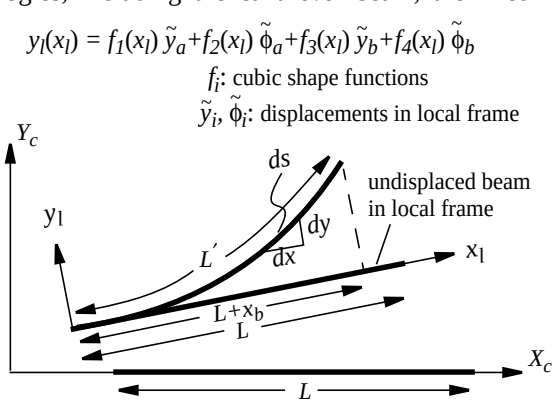

Figure 4: Calculation of effective beam length.

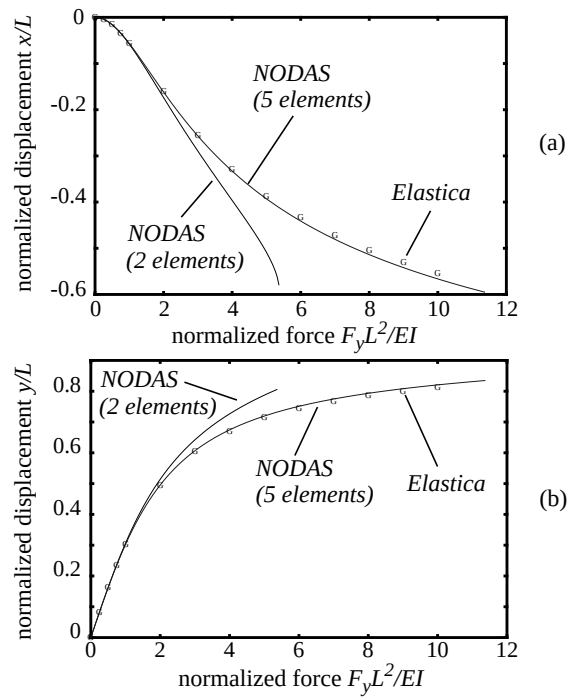

Figure 5: Static analysis of a cantilever beam with *Fy* applied to the free end (100 µm long, 2 µm wide, 2 µm thick, *E*=165GPa). (a) normalized *x* displacement (b) normalized *y* displacement.

beam, the crab-leg spring and the folded-flexure spring, have been simulated and compared with FEA and exact analytical results.

Figure 5 shows the static analysis of a cantilever beam with a force applied in the *y*-direction, modeled with two and five beam elements respectively. The normalized displacements vs. normalized force *Fy* are plotted and compared to the elastica solution. With two beam elements, large deflections up to 50% of the beam length are modeled with an error less than 5%. Using five beam elements, the applicable range is increased to more than 80% of the beam length. The simulation (DC sweep of *Fy*) takes about 20 s on a 450 MHz Sun workstation.

Figure 6 shows the static analysis of a fixed-fixed beam with a central concentrated load, *Fy*, modeled with two, four and ten beam elements. Results are compared to FEA simulation in ABAQUS. Because the force is applied to the middle of the beam, at least two beam elements are needed for the entire simulation, one for each half. The normalized axial force in the fixed-fixed beam starts to be on the order of mN at a displacement of 5 µm. The nonlinearity starts to be evident at a very small lateral displacement of 1 µm. Displacement is very small even for a very large normalized lateral force of 27 (*i.e.*, 600 µN). As the lateral displacement is small, even two elements provide very accurate results with the error less than 5%. Using four and ten elements further reduce the error to within 2% and 1%, respectively. Similar simulation accuracies are obtained for crab-leg springs and folded-flexure springs.

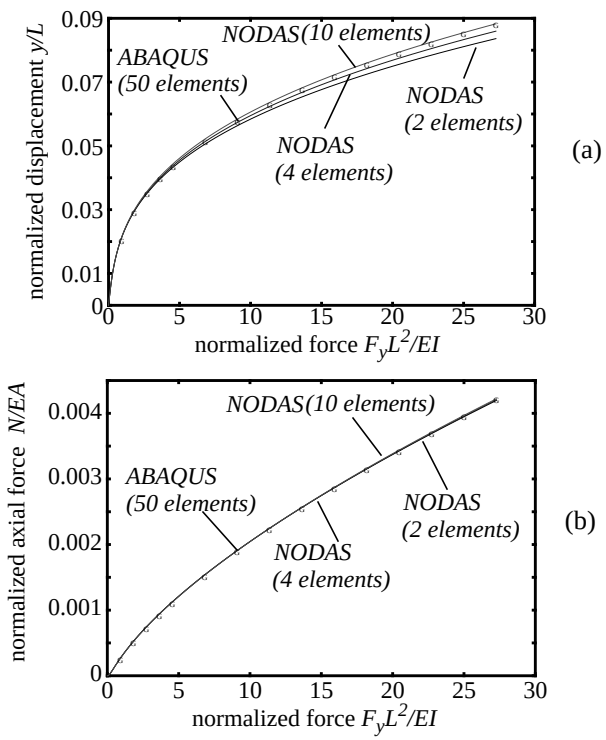

Figure 6: Static analysis of a fixed-fixed beam with *Fy* applied to the middle (100 µm long for each side, 2 µm wide, 2 µm thick, *E*=165GPa). (a) normalized *y* displacement (b) normalized axial force.

A folded-flexure resonator excited by a large sinusoidal force is simulated to verify existence of the duffing effect, a well-known phenomenon caused by beam nonlinearity. The amplitude of the force is set to be such that the displacements are small at low frequencies but are large enough to cause the nonlinearity at frequencies near to the resonance. Figure 7 shows the steady-state envelope of the displacement magnitude obtained from a series of transient analysis with varying excitation frequencies. Results given by the linear and nonlinear beam models are compared. The frequency shift due to the beam stiffening is clearly shown. The transient simulation with a simulation time of 1 ms (~20,000 time steps) takes about 10 min when the frequency

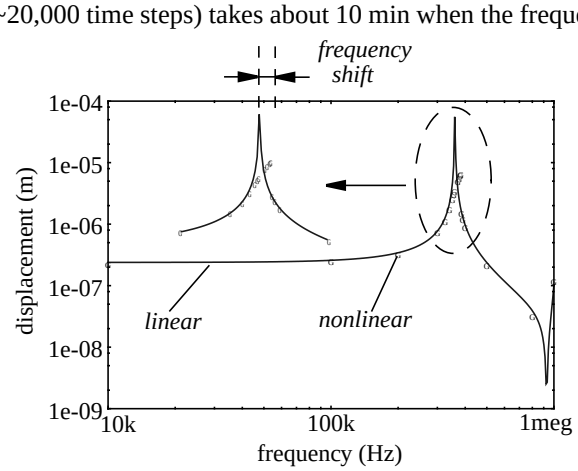

Figure 7: Duffing effect simulation.

falls outside of the resonance, and about 1 hr when the frequency is very close to the resonance.

CONCLUSIONS

The nonlinear beam behavioral model presented captures the geometric nonlinearity caused by large axial stress and large deflection. It gives excellent simulation accuracy for very large deflections and axial stresses, when about five beam elements are used. The excellent accuracy of the model positively impacts the general applicability of composable design of MEMS using commercial electronic design tools.

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