Mixing and Impulse Extremization in Microscale Vortex Formation

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ABSTRACT

A key idea in microscale mixing, where the conventional turbulent mixing is not applicable, is to increase the area of the interface between two fluids. Significant entrainment and consequently mixing during vortex formation works based on the same idea of increasing the interface between two fluids. In this study the previous results by Mohseni *et al.* [1–4] in universality of vortex formation is extended to optimize microscale mixing or impulse generation. The energy, circulation, and impulse of the optimal vortex rings are calculated by the direct numerical simulation of the Navier-Stokes equations. A discussion of the governing parameters are presented. There are only two nondimensional parameters that governs the vortex pinch off process: the nondimensional energy and the nondimensional circulation. Mixing and impulse generation are also characterized by the value of these nondimensional numbers.

Keywords: Vortex, mixing, microfluidics, impulse, universality.

1 INTRODUCTION

Microscale mixing is of crucial importance in many branches of the new emerging micro/nanoscale technologies. Conventionally, mixing of fluids involves generating turbulence. At small length scales, however, such traditional methods are extremely hard to achieve yet diffusion is often too slow. A key concept is that an increase in the area of the interface between two fluids corresponds to increased mixing. We use this concept to increase the interface between two fluids by the formation of vortices at the interface. Vortex formation involves significant entrainment and results in efficient mixing.

Recently, microscale vortex ring formation in air was used to generate lifting impulse in a microairborn platform (see Muller *et al.* [5, 6] and references in there). A challenging task in such studies is to characterize the parameters of a vortex generator to produce maximum impulse. In this study we characterize optimal vortex formation for maximum mixing or impulse generation.

The generation, formation, evolution, and interactions of vortex rings have been the subject of numerous studies (*e.g.* [7]). The total entrainment (or mixing) in a vortex depends strongly on the formation process. To this end, we focus on the universal formation number of vortex ring pinch-off process observed initially in experiments by Gharib *et al.* [8], and later on a theoretical modeling by Mohseni & Gharib [1], and numerical simulations of the Navier-Stokes equations by Mohseni *et al.* [2]. In the laboratory, vortex rings can be generated by the motion of a piston pushing a column of fluid through an orifice or nozzle. The boundary layer at the edge of the orifice or nozzle will separate and roll up into a vortex ring. We think that since the formation of vortex rings involves strong mixing of the generated shear layer with the ambient fluid, the ergodicity requirement of statistical equilibrium theory has a chance to be satisfied. The experiments of Gharib *et al.* [8] have shown that for large piston stroke versus diameter ratios (L/D), the generated flow field consists of a leading vortex ring followed by a trailing jet. In figure 1 the time evolution of vorticity contours in a pinched-off vortex are depicted from the direct numerical simulations of the Navier-Stokes equations. The vorticity field of the formed leading vortex ring is disconnected from that of the trailing jet at a critical value of L/D (dubbed the "formation number"), at which time the vortex ring attains a maximum circulation. The formation number was in the range 3.6 to 4.5 for a variety of exit diameters, exit plane geometries, and non-impulsive piston velocities. An explanation for this phenomenon was given based on Kelvin's variational principle. It was both experimentally [8] and analytically [1] observed that the limiting stroke L/D occurs when the generating apparatus is no longer able to deliver energy, circulation and impulse at a rate compara-

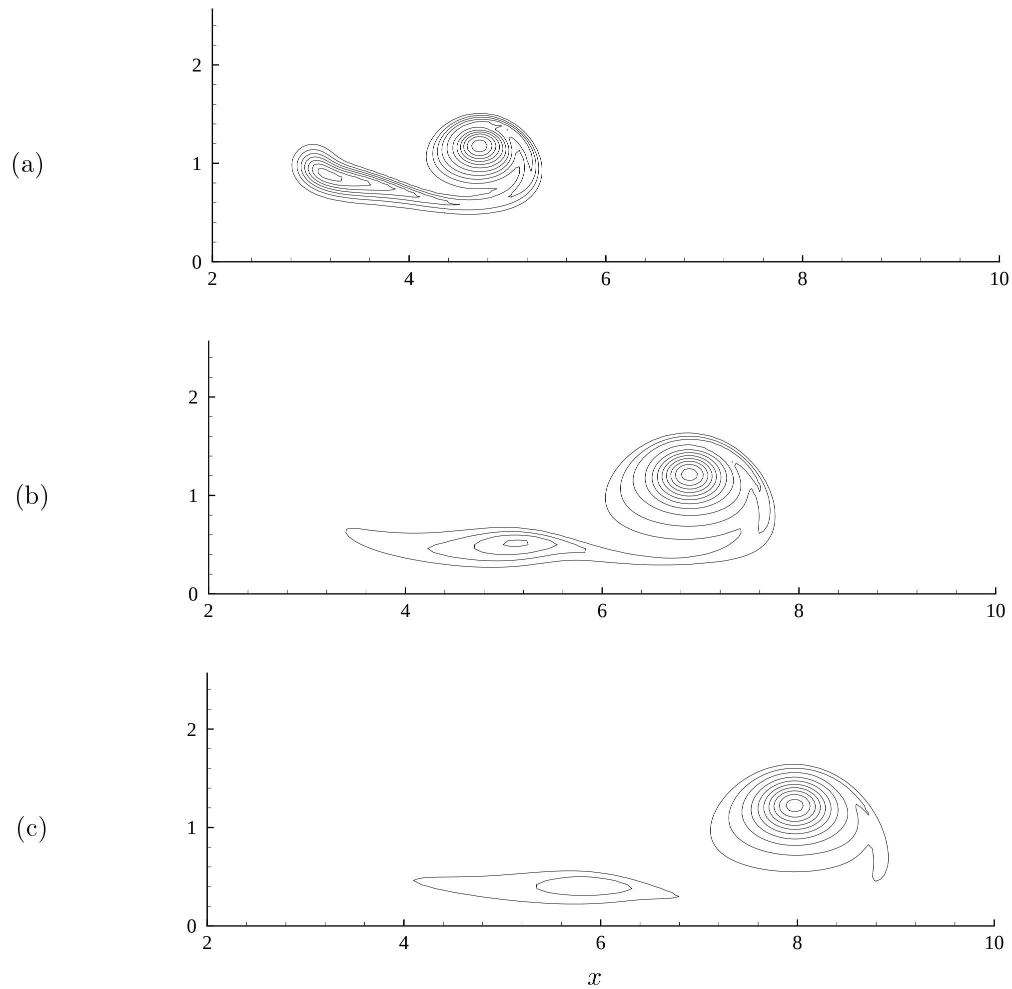

Figure 1: Time evolution of vorticity contours in a direct numerical simulation of the vortex ring pinched-off Process. The detachment of the trailing jet from the leading vortex ring is evident in (c). Horizontal and vertical axes are x and r respectively.

ble with the requirement that a steadily translating vortex ring has maximum energy with respect to kinematically allowable perturbations. As demonstrated in [3], Kelvin's variational principle (energy extremization) has a close connection with the entropy maximization in statistical equilibrium theory. Numerical evidence for a relaxation process to an equilibrium state has already been provided by Mohseni *et al.* [2] in a direct numerical simulation of the pinch-off process. Inspired by these observations Mohseni offered a relaxational (statistical) approach to the pinch-off process [1, 3]. This is an alternative explanation of the vortex ring pinch-off process, based on a mixing entropy maximization, besides the energy extremization approach in Kelvin's variational principle: The system relaxes to an equilibrium state from its initial configuration dictated by the stroke ratio L/D in the cylinder-piston mechanism of vortex ring generation or more generally by TU/h for a general vortex ring generator. Here, T is the period of vortex shedding, U is the translational velocity, and h is the toroidal radius of the resulting vortex ring. It was suggested that the final state of the problem is governed by the first few invariants of motion, namely the energy, impulse, and circulation. Note that any vorticity generation mechanism has its own specific rate for the generation of theses invariants of motion. For a cylinder piston mechanism these rates are given in Mohseni & Gharib [1]. The physical explanation is that for short strokes the system relaxes to a small steadily translating vortex ring. Increasing the stroke ratio results in a larger vortex ring. For high enough strokes (above the formation number) the traditional cylinder piston mechanism is not able to provide energy compatible with an equilibrium state at the same circulation and impulse that maximizes the mixing entropy in the statistical equilibrium theory. This is an alternative explanation, besides the energy extremization in Kelvin's variational principle, for the vortex ring pinch-off process.

2 MODELING OPTIMIZED VORTEX RING FORMATION

Consider an arbitrary generator of axisymmetric vortex sheets. A popular example is the cylinder-piston mechanism, where a vortex sheet is ejected at a particular speed, roughly the piston velocity. It is known that any cylindrical vortex sheet is unstable and will role up to a vortex ring. While the vortex ring is forming it will accelerate due to its induction velocity. This combination of vortex ring enlargement and acceleration continues until the vortex sheet is unable to feed any more vorticity to the vortex ring. In that case the leading vortex ring will impart from the vortex sheet and the remaining vortex sheet goes through the instability process again to form a new vortex ring. Continuation of this process in axisymmetric flows will eventually results in the formation of a periodic array of vortex rings with toroidal diameter h and spacing l. Each vortex will move with a translational velocity U_{tr}. The period of each vortex ring shedding will, then, will be $t = l/U_{tr}$. In such a process the parameters involved are the geometrical parameters l and h, the translational velocity U_{tr}, and the main invariants of motion: Energy E, impulse I, and total circulation Γ (in Euler equations). One can expect a functionality of the form $f(l, h, U_{tr}, E, I, \Gamma)$. A straightforward dimensional analysis results in

$$t_{nd} = g(E_{nd}, \Gamma_{nd}) \tag{1}$$

where

$$t_{nd} = \frac{l}{h} = \frac{tU_{tr}}{h}, \quad \text{nondimensional formation time,} \tag{2}$$

$$E_{nd} = \frac{E}{\Gamma^{3/2} I^{1/2}}, \quad \text{nondimensional energy,} \tag{3}$$

$$\Gamma_{nd} = \frac{\Gamma}{I^{1/3} U_{tr}^{2/3}}, \quad \text{nondimensional circulation.} \tag{4}$$

These nondimensional numbers govern the vortex ring pinch-off process.

There are various experimental, numerical and engineering methods to generate vortex rings. Each of them has its own specific rate of generation of invariants of motion. In order to predict the limiting formation number for each vortex generator one needs to estimate the invariants of motions for that particular vortex ring generator. The nondimensional energy (3) and circulation (4) are then formed and equated to an estimate of the same quantities for the resulting pinched-off vortex. In a series of direct numerical simulation of the vortex pinch-off process, Mohseni *et al.* [2] showed that Norbury family of vortices [9] provide an accurate approximation of the invariants of motion of the pinched-off vortex ring.

The values of the nondimensional energy and circulation for the pinched-off leading vortex were predicted in [1]. The predicted value of E_{nd} for the pinched-off leading vortex ring is approximately 0.3, while the limiting value for the non-dimensional circulation Γ_{nd} is approximately 2. These predictions were also verified in direct numerical simulation of the vortex ring pinch-off process [2].

3 NONDIMENSIONAL IMPULSE

In the previous section we showed that the main nondimensional numbers governing the vortex ring pinch-off process are t_{nd}, E_{nd}, and Γ_{nd}. The impulse I appears explicitly only in E_{nd} and Γ_{nd}. Consequently, one can rearrange these quantities to define two new nondimensional impulses:

$$I^E_{nd} = \frac{I\Gamma^3}{E^2} = \frac{1}{E^2_{nd}} \tag{5}$$

$$I^\Gamma_{nd} = \frac{IU^2_{tr}}{\Gamma^3} = \frac{1}{\Gamma^3_{nd}} \tag{6}$$

Therefore, extremization of impulse depends on the formation process. For a vortex ring generator that delivers the same amount of kinetic energy and circulation, the pinched-off vortex ring has the extremum impulse I^E_{nd}. This number is estimated to be around $I^E_{nd} = 1/E^2_{nd} = 1/0.3^2 \approx 11$. On the other hand for a vortex ring generator that delivers the same amount of circulation with fixed translational velocity, a pinched off vortex produces the maximum impulse I^Γ_{nd}. This number can be estimated to be around $I^\Gamma_{nd} = 1/\Gamma^3_{nd} = 1/2^3 \approx 0.12$.

As an example the nondimensional impulse I^Γ_{nd} is directly calculated in a series of direct numerical simulations. The details of the numerical techniques and parameters of each simulation are tabulated in [2]. Figure 2 shows the computed values

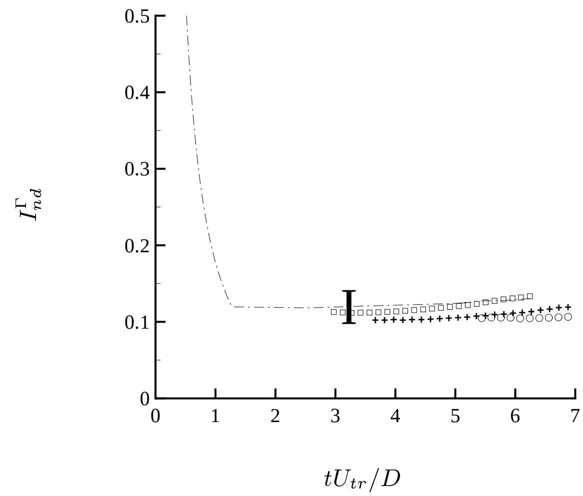

Figure 2: Vortex rings with different forcing duration. Normalized impulse, I^{Γ}_{nd}. The line corresponds to the total impulse in the domain, while the symbols correspond to impulse of the leading vortex ring. $TC/R^2 = 9.11$, Case 3 (—·—); $TC/R^2 = 14.2$, (□); $TC/R^2 = 25.3$, (+); $TC/R^2 = 39.5$, (∘). See Mohseni *et al.* [2] for the notation.

of I^{Γ}_{nd} for these cases. It is evident from this figure that the value of the nondimensional impulse I^{Γ}_{nd} for the leading vortex ring matches accurately with our theoretical prediction of $I^{\Gamma}_{nd} \approx 0.12$.

4 CONCLUSIONS

It is believed that that the goodness of mixing achieved in dominantly laminar-flow processes in microscale devices depends on the net amount of shear supplied to the domain (*e.g.* [10, 11]). Shear increases the interface area between the two fluids. To this end, it is argued that vortex formation could significantly enhance microscale mixing. An important question is, then, how to generate a vortex with maximum mixing properties (or maximum entrainment). A larger vortex ring entrains more ambient fluids and results in increased mixing. A vortex ring with maximum nondimensional circulation is achieved at the formation number of 4. The corresponding value of the nondimensional circulation of the leading vortex ring for best mixing is estimated to be around 2.

Another recent application of vortex ring generation at microscale is thrust generation [5, 6] in micro synthetic jets. We believe that the optimal vortex rings in this case are obtained for the formation number 4. The appropriate scalings for the impulse generation by vortex rings are found and the optimal values of the nondimensional impulse are found to be around $I^{E}_{nd} = I\Gamma^3/E^2 \approx 11$, and $I^{\Gamma}_{nd} = IU^2_{tr}/\Gamma^3 = 1/8$.

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