Nonlinear Analysis of Electrostatic Actuation in MEMS with Arbitrary Geometry

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ABSTRACT

We present an explicit formulation for the Jacobian (or tangent) matrix of the discretized non-linear model for a coupled electromechanical system. The Jacobian matrix, consisting of the derivatives of the residual, is needed in Newton-Raphson-based algorithms to solve these non-linear problems, as well as in the analysis of their stability. Our formulation of the Jacobian matrix relies on the discretization of the electrostatic sub-model by means of the collocation boundary-element method. It is independent of the discretization method (either boundary or finite elements) and of possible non-linearities in the mechanical sub-model.

1 INTRODUCTION

The electrostatic actuation of a mechanical system is a coupled energy-domain problem, where the mechanical deformation and the charge distribution depend on each other. These transducers play a dominant role in microsystem engineering as new applications, designs and technology are developed and improved. Examples are airbag-accelerometers, filters, and deflectable micromirror arrays. The issues raised in their modeling are due to the intrinsic non-linearity introduced by the coupling. Indeed, even for the case of small deformations and displacements, when the mechanical system alone could be safely considered linear, the electrostatic pressure acting on conducting surfaces is proportional to the square of the charge density, and, second, the capacitance matrix, to be multiplied by the charge density, depends on the mechanical deformation.

The electrostatic potentials of the conductors are the driving signals for the actuator, and can therefore be considered as parameters when analyzing the actuator as an independent system component. Besides the obvious problem of determining the configuration of the system (in terms of displacement and charge distribution) corresponding to a certain input, limit and bifurcation points are relevant issues in the design of the system, as they can lead to instabilities in the behavior of the system. All these problems demand non-linear solution algorithms, for which relaxation methods have been proposed [1]. Most of contemporary research on non-linear analysis has focused on homotopy methods that have Newton-like iterations at their core: the residual of an equation is linearized around the tentative solution to obtain its increment. In formulae, given the residual equation

Linear	Non-Linear
$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$	$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}}{2}$
$f_j = \partial_i \sigma_{ij}$	$f_j = \partial_i(\sigma_{ij} + \sigma_{ik}u_{j,k})$

Table 1: Strain and stress-induced force for linear and non-linear mechanics

$$R(w) = 0 \tag{1}$$

a Newton iteration consists in solving the linearized problem

$$[DR(w)](w - w') = R(w) \tag{2}$$

for w', with w known. The matrix $[DR(w)]_{ij} = \partial R_i / \partial w_j$ is the Jacobian matrix and consists of the derivatives of the residual with respect to the unknowns. Matrix-free methods provide an iterative solution to equation (2), where at every step the product of the Jacobian matrix times the linear solution update is estimated [2]. In this approach no explicit knowledge of the Jacobian matrix is needed. We follow a different approach, taking advantage of the knowledge of the coupling that is available. Reprising other work [3], we give an explicit formulation of the Jacobian matrix. It allows the matrix to be assembled and therefore stored, to be reused in modified Newton algorithms and eigenvalue problems for stability analysis.

2 PROBLEM FORMULATION

The systems considered in this paper consist of a set of perfect conductors embedded in dielectric materials. The conducting surfaces experience an electrostatic pressure

$$P = -\chi(\nabla\varphi)^2 \tag{3}$$

where χ is the dielectric constant and φ is the potential at the interface on the side of the dielectric. In turn, the electrostatic potential satisfies the Poisson equation

$$\nabla^2\varphi = 0 \tag{4}$$

defined on the region occupied by the dielectric Ω_ε. The electrostatic problem is defined on the geometry resulting from deforming the initial configuration according to the displacement field $\boldsymbol{u}(\boldsymbol{x})$. Depending on whether a linear or nonlinear mechanical model is selected, the strain ε_{ij} and the stress-induced force density $\boldsymbol{f}$ are taken from the corresponding column of table 1. In both cases we assume the materials to be linear elastic by expressing the relation between stress σ_{ij} and strain in terms of the of the fourth-order elastic tensor E_{ijkl} as

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \tag{5}$$

In the absence of body forces, the mechanical partial-differential equation (PDE) reduces to

$$\boldsymbol{f} = 0 \tag{6}$$

The electrostatic pressure poses the additional constraint

$$(\sigma_{ij} - P\delta_{ij})n_j = 0 \tag{7}$$

on the conductor surfaces. The normal $\boldsymbol{n}$ points towards the outside of the conductor. If the conductor surface is on the boundary of the domain of definition Ω of equation (6), then constraint (7) represents a natural boundary condition, whilst if the surface is located inside the domain then the constraint represents a surface source for equation (6). This difference is of very little importance on the practical side, as after discretization both sources and natural boundary conditions form the known terms of the equation system. Following standard variational techniques, scalar multiplication of the force density by any kinematically admissible variation $\delta\boldsymbol{u}$ of the displacement, followed by integration over the domain and application of the Gauss-Green formula, we obtain the equation of virtual work

$$\int_\Omega \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}^3x = \int_\Gamma P n_i \delta u_i \,\mathrm{d}S(x) \tag{8}$$

where Γ is the union of the conductor surfaces.

3 DISCRETIZATION

Both finite and boundary elements are suitable for discretizing the variational equation (8). Both methods interpolate the displacement using vector shape functions $\boldsymbol{M}_j(\boldsymbol{x})$ (typical of the method) with coefficients u_j as

$$\boldsymbol{u}(\boldsymbol{x}) = \sum_{j=1}^{m} u_j \boldsymbol{M}_j(\boldsymbol{x}) \tag{9}$$

to produce an equation for each nodal force component $T_i(u_1 \ldots u_m)$. The final outcome is an equation system of the form

$$T_i(u_1 \ldots u_m) = \int_\Gamma \boldsymbol{w}_i \cdot \boldsymbol{n} P \,\mathrm{d}S(x) \tag{10}$$

The left-hand side expresses the elastic reaction force and is independent of the charge distribution. The right-hand side expresses the electrostatic force in terms of a weighting function $\boldsymbol{w}_i$. For the Finite Element Method (FEM) $\boldsymbol{w}_i$ is equal to the shape function $\boldsymbol{M}_i$. For the Boundary Element Method (BEM) $\boldsymbol{w}_i$ is equal to the elastic fundamental solution centered at the i-th collocation node. We interpolate the derivative of the potential normal to the surface, signified by q, using a set of discontinuous shape functions N_j as

$$q(\boldsymbol{x}) = \sum_{j=1}^{n} q_j N_j(\boldsymbol{x}) \tag{11}$$

Using (11), the force-balance equation (10) is rewritten as

$$T_i(u_1 \ldots u_m) = \sum\nolimits_{jk} C_{ijk} q_j q_k \tag{12}$$

where the new set of coefficients C_{ijk} is defined by

$$C_{ijk} = -\int_\Gamma \chi \boldsymbol{w}_i \cdot \boldsymbol{n} N_j N_k \,\mathrm{d}S(x) \tag{13}$$

To use the BEM to discretize the electrostatic problem, the Poisson equation (4) is first recast as a Fredholm integral equation of the first kind [4]. Assuming an undeformed geometry we obtain

$$V(\xi) = \int_{\partial\Omega_\varepsilon} \Phi(\xi - \boldsymbol{y}) q(\boldsymbol{y}) \,\mathrm{d}S(\boldsymbol{y}) \qquad \boldsymbol{x} \in \Omega_\varepsilon \tag{14}$$

where V is the potential on the conductors and Φ is the Poisson fundamental solution. Equation (14) is collocated at a set of points ξ_i (each at potential V_i), yielding the discrete equation system

$$V_i = \sum\nolimits_j q_j \int_{\partial\Omega_\varepsilon} \Phi(\xi_i - \boldsymbol{y}) N_j(\boldsymbol{y}) \,\mathrm{d}S(\boldsymbol{y}) \tag{15}$$

A mechanical deformation of the system affects the integral on the right-hand side of equation (15) in two ways. The collocation node ξ_i is displaced by $\boldsymbol{u}(\xi_i)$ and the domain of integration undergoes a change of coordinate

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}) \tag{16}$$

Equation (15) is reformulated as

$$V_i = \sum_j S_{ij}(u_1, \ldots, u_m) q_j \tag{17}$$

where the inverse capacitance matrix S is defined by

$$\begin{aligned} &S_{ij}(u_1, \ldots, u_m) = \\ &= \int_{\partial\Omega_\varepsilon} \Phi(\xi_i + \boldsymbol{u}(\xi_i) - \boldsymbol{y}) N_j(\boldsymbol{y}) \det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right) \mathrm{d}S(\boldsymbol{x}) \end{aligned} \tag{18}$$

The residuals of the mechanical and electrostatic equations are defined as

$$(R_M)_i = \sum_{jk} C_{ijk} q_j q_k - T_i(u_1, \ldots, u_m), \quad i = 1 \ldots m \tag{19}$$

$$(R_E)_i = V_i - \sum_j S_{ij}(u_1, \ldots, u_m) q_j, \quad i = 1 \ldots n \tag{20}$$

respectively. Combining the degrees of freedom into the vector $w = (u, q)$ and the residuals into $R = (R_M, R_E)$, the equation system has the form of (1), and its tangent matrix has four blocks:

$$\frac{\partial R}{\partial w} = \begin{bmatrix} \partial R_M / \partial u & \partial R_M / \partial q \\ \partial R_E / \partial u & \partial R_E / \partial q \end{bmatrix} = \begin{bmatrix} -K & F \\ G & S \end{bmatrix} \tag{21}$$

The top-left block K is the differential mechanical stiffness, whose formulation is well-known [5]. Under the assumptions of linear mechanics, it coincides with the structural stiffness. The bottom-right block is the inverse capacitance matrix (18). The top-right block is simply

$$F_{ij} = 2 \sum_k C_{ijk} q_k \tag{22}$$

The bottom-left block, whose components are given by

$$G_{ij} = \sum_k \frac{\partial S_{ik}}{\partial u_j} q_k \tag{23}$$

requires an explicit formulation of the derivative of the integral (18).

4 THE TANGENT MATRIX

The integral (18) can be restricted to the support of N_j which coincides with one of the elements, say Γ_j. Γ_j is mapped to a reference element Q_j, typically the triangle

$$\{(z_1, z_2) \in R^2 : z_1 \geq 0, z_2 \geq 0, z_1 + z_2 \leq 1\} \tag{24}$$

or the quadrilateral

$$\{(z_1, z_2) \in R^2 : |z_1| \leq 1, |z_2| \leq 1\} \tag{25}$$

Assuming an undeformed geometry, the function mapping the reference element Q_j to the actual element Γ_j is a convex combination of the vertices ζ_k of Γ_j so that

$$\boldsymbol{x} = \psi_j^0(\boldsymbol{z}) = \sum_k m_{jk}(\boldsymbol{z}) \zeta_k \tag{26}$$

The summation in (26) is actually extended to all mesh nodes imposing m_{jk} to be zero if node k is not a vertex of the element Γ_j. The mechanical deformation adds a perturbation

$$\delta\psi_j(\boldsymbol{z}, u_1, \ldots, u_m) = \sum_{kh} u_h \boldsymbol{M}_h(\zeta_k) m_{jk}(\boldsymbol{z}) \tag{27}$$

to the mapping, which becomes

$$\boldsymbol{y} = \psi_j(\boldsymbol{z}, u_1, \ldots, u_m) = \psi_j^0(\boldsymbol{z}) + \delta\psi_j(\boldsymbol{z}, u_1, \ldots, u_m) \tag{28}$$

The argument of the fundamental solution in (18) is written as

$$\xi_i + \boldsymbol{u}(\xi_i) - \boldsymbol{y} = \xi_i - \psi_j^0(\boldsymbol{z}) + \sum_h \Delta_{ijh}(\boldsymbol{z}) u_h \tag{29}$$

where the perturbation coefficients are given by

$$\Delta_{ijh}(\boldsymbol{z}) = \boldsymbol{M}_h(\xi_i) - \sum_k \boldsymbol{M}_h(\zeta_k) m_{jk}(\boldsymbol{z}) \tag{30}$$

The capacity matrix elements are then expressed in the reference space as

$$S_{ij} = \int_{Q_j} \Phi(\xi_i - \psi_j^0 + \sum_h \Delta_{ijh} u_h) N_j(\psi_j) J_j \, \mathrm{d}^2 z \tag{31}$$

The composition $N_j(\psi_j(\boldsymbol{z}))$ expresses the shape function $\hat{N}_j(\boldsymbol{z})$ in the reference element, which is independent on the configuration of the element, and consequently from the displacement. The Jacobian $J = \|\psi_{,1} \times \psi_{,2}\|$ depends on the displacement via the perturbation (27). Since Q_j is fixed, the derivative in (23) can be carried inside the integral (31), and,

of the three factors under integration, only $\Phi(\ldots)$ and J give non-zero derivatives. The former is

$$\frac{\partial \Phi(\xi_i - \psi_j^0 + \sum_h \Delta_{ijh} u_h)}{\partial u_k} = \nabla \Phi(\ldots) \cdot \Delta_{ijk} \tag{32}$$

whereas for the latter the transport theorem for shells [6] gives

$$\frac{\partial J_j(\boldsymbol{z}, u_1 \ldots u_m)}{\partial u_k} = J_j(\boldsymbol{z}) L_{jk}(\boldsymbol{z}) \tag{33}$$

$$L_{jk} = (1 - \boldsymbol{n}_j \boldsymbol{n}_j) \cdot \left(\frac{\partial \psi_j}{\delta \boldsymbol{z}} \right)^{-1} \cdot \sum_h m_{jh} \boldsymbol{M}_k(\zeta_h) \tag{34}$$

Summing the two contributions yields

$$\begin{aligned} \frac{\partial S_{ij}}{\partial u_k} &= \int_{Q_j} \nabla \Phi(\xi_i - \psi_j^0 + \textstyle\sum_h \Delta_{ijh} u_h) \cdot \Delta_{ijk} \hat{N}_j J_j \, \mathrm{d}^2 z \\ &+ \int_{Q_j} \Phi(\xi_i - \psi_j^0 + \textstyle\sum_h \Delta_{ijh} u_h) \hat{N}_j L_{jk} J_j \, \mathrm{d}^2 z \end{aligned} \tag{35}$$

5 CONCLUSIONS

The electromechanical coupling involves a tangent matrix comprising four sub-matrices (21). The two diagonal blocks K and S are exactly the conventional mechanical stiffness and the inverse capacitance. Since S is already required for the evaluation of the residual, its assembly does not add to the computational complexity of the method. For linear mechanics K is also needed for the residual. For non-linear mechanics, the assembly of the stiffness matrix would also be necessary to solve the coupled problem using a relaxation method. One of the two off-diagonal blocks, F, involves terms that are already required for the computation of the residual. The dependence of F on a hyper-matrix with three indices C_{ijk} does not induce a cubic complexity. It presents the same sparsity and computational effort of the stiffness matrix, due to the fact that the product $N_j N_k$ in (13) vanishes if the nodes j and k are not on the same element. This is consistent with the local nature of the electrostatic pressure which has a point-wise dependence on the electrostatic field q. The other off-diagonal block G is expressed by a linear combination of two integrals that have the same form as the single and double layer potentials that are used in assembling standard BEM equation systems. Consequently, the same methods that are used to sparsify their structure and accelerate their computation are applicable.

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