Calculation of the Electroquasistatic Sinusoidal Steady-State Coulomb Force on a Conductor Coated with a Lossy Dielectric

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ABSTRACT

The ability to directly calculate the analytical closed-form electroquasistatic sinusoidal steady-state Coulomb force (or ac force) exerted on a conductor coated with a lossy dielectric in the presence of a second grounded conductor and excited by a sinusoidal voltage is of interest in areas such as Microelectromechanical Systems (MEMS). This ac force is due to the phenomenon of dielectric relaxation existing in the lossy dielectric and is dependent on the operating frequency of the voltage source as well as the material properties of the lossy dielectric. In this paper, the authors apply a new technique based on expressing the Coulomb force in terms of a special singularity integral to *directly* calculate the total ac force exerted on a conductor coated with a lossy dielectric. Specifically, the example of the calculation of the Coulomb ac force on a simplified structure consisting of a solid incompressible lossy dielectric slab attached to the lower plate of a parallel-plate capacitor excited by a sinusoidal voltage is considered.

***Keywords*:** Coulomb force, time-average sinusoidal steady-state force, lossy dielectric, dielectric relaxation, MEMS.

1 INTRODUCTION

In the field of Microelectromechanical Systems (MEMS), the ability to directly calculate the analytical closed-form time-average electroquasistatic sinusoidal steady-state Coulomb force (or ac force) exerted on a conductor coated by a lossy dielectric in the presence of a second grounded conductor excited by a sinusoidal voltage is useful and has relevant applications. For example, it may be required to calculate the ac force on a highly-doped silicon cantilever beam MEMS structure coated with a lossy dielectric oxide layer suspended over a parallel ground plane with a sinusoidal voltage applied between the beam and the ground plane.

Recently, a new straightforward technique was proposed [1] to directly calculate electromagnetic energy in idealized lossless electrical circuit problems. In this technique, the electromagnetic energy under consideration is expressed in terms of a special singularity integral identity which consists of the integral of the product of the unit impulse function $\delta(x)$ and the associated unit step function $u(x)$ [2] given in its most general terms by:

$$\int_{-\infty}^{\infty} u(x)\delta(x)\,dx = \tfrac{1}{2} \qquad (1)$$

where $\delta(x-a)=du(x-a)/dx$. The formal derivation of (1) is given in [1] on the basis of the associated work by Bracewell [2]. Note that this new technique is only applicable in the interesting class of problems where the quantity of interest (e.g., energy) can be expressed directly in terms of (1) (i.e., the integral of the product of an impulsive quantity and its associated step-varying quantity). In this paper, we *extend* [1] to the direct and straightforward calculation of electroquasistatic ac force.

For a simple example of the use of this new technique for force calculation, let us first consider the simpler calculation of the analytical closed-form electrostatic Coulomb force (or dc force) exerted on the lower plate of the simple parallel-plate capacitor shown in Fig. 1 consisting of two perfectly conducting plates ($\sigma=\infty$) each of area A located at $x=0$ and $x=d$ and driven by a dc voltage source V_0. Fringing fields are neglected. The region between $x=0$ and $x=d$ is assumed to be free space represented by ε_0. This voltage produces a normal uniform x-directed dc electric field E_x in the vacuum region between the plates of the capacitor. The nature of the induced dc force will be *attractive* between the upper and lower conducting plates. Using the new technique described in this paper, this dc force can be calculated *directly* by expressing the Coulomb force integral equation [3] in terms of the singularity integral identity given by (1). Specifically, the total Coulomb dc force acting on the plate at $x=d$ can be expressed in terms of (1) as the integral evaluated at $x=d$ of the product of the equivalent infinitely thin, impulsive volume charge density ρ_v (representing the surface charge density σ_s at $x=d$) and the associated electric field E_x which has a step-variation. Using the appropriate Gaussian boundary conditions at $x=d$, we can express $\rho_v=\sigma_s\delta(x-d)$, $\sigma_s=-\varepsilon_0E_0$, $E_x=E_0[1-u(x)]$ and $E_0=V_0/d$. There-fore, the total Coulomb dc force F_x induced by E_x acting on the total charge Q (due to ρ_v) residing on the lower plate of volume V can be calculated as follows:

$$
\begin{aligned}
F_x &= \int_{Q_{\text{lower-plate}}} E_x\,dq = \int_{V_{\text{lower-plate}}} \rho_v E_x\,dv \\
&= \int_{d^-}^{d^+} \underbrace{\sigma_s \delta(x-d)}_{\rho_v} \underbrace{E_0\left[1-u(x)\right]}_{E_x} A\,dx \\
&= -\frac{\varepsilon_0 V_0^2 A}{d}\left[\underbrace{\int_{d^-}^{d^+}\delta(x-d)dx}_{1} - \underbrace{\int_{d^-}^{d^+}\delta(x-d)\,u(x-d)dx}_{1/2}\right] \\
&= -\frac{\varepsilon_0 V_0^2 A}{2d^2}
\end{aligned}
\tag{2}
$$

This result agrees with the literature [3, 4]. Notice that (1) is the source of the essential factor of "1/2" in (2) which is inherent in this class of problems. This example demonstrates the essence of the new technique described in this paper.

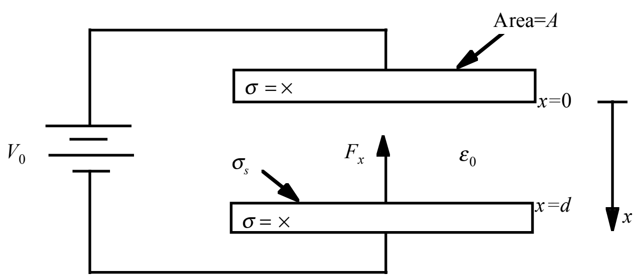

Figure 1: Parallel-plate capacitor excited by a dc voltage.

The authors extend this new technique to directly calculate the ac force exerted on a dielectric-coated conductor where the dielectric is assumed to be lossy. Specifically, the example of the calculation of the ac force exerted on the simplified structure consisting of a solid incompressible lossy dielectric slab attached to the lower conducting plate of a parallel-plate capacitor excited by a sinusoidal voltage source is considered.

2 CALCULATION OF THE AC FORCE

Consider the parallel-plate capacitor shown in Fig. 2 consisting of two perfectly conducting plates (σ=∞) each of area A located at x=0 and x=d. A solid incompressible lossy dielectric slab of thickness $t < d$, conductivity σ and permittivity $\varepsilon = \varepsilon_r \varepsilon_0$ is attached to the lower plate of the capacitor. The region between x=0 and $x=a=d-t$ is assumed to be free space represented by ε_0. A sinusoidal voltage $V_0\cos(\omega t)$ is applied across the conducting plates of the capacitor. This voltage produces normal uniform sinusoidal steady-state x-directed electric fields in both the free space and the dielectric material regions between the plates of the capacitor. It is assumed that the physical dimensions of the capacitor are much less than a wavelength λ (where $\lambda = 2\pi c/\omega$ and c is the speed of light in the dielectric material) such that the electroquasistatic approximation is valid [5]. We seek to calculate the total time-average analytical closed-form Coulomb ac force $\langle \underline{F}_x \rangle$ exerted on the lower dielectric-coated conductor plate. Note that we will use standard sinusoidal steady-state phasor notation for this analysis.

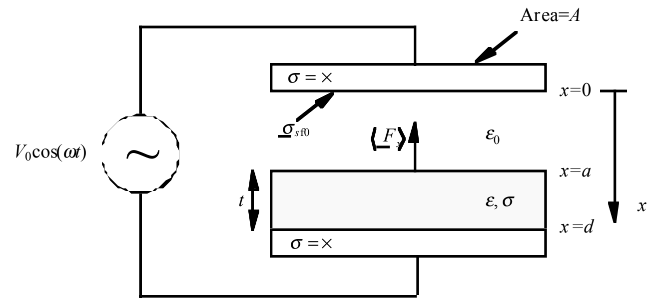

Figure 2: Parallel-plate capacitor with solid incompressible lossy dielectric slab attached to the lower electrode and excited by sinusoidal voltage source.

Neglecting fringing fields, the total electric field phasor $\underline{E}_x$ existing between the capacitor plates can be written in terms of a pair of unit step functions as:

$$
\underline{E}_x = \underline{E}_0\left[u(x) - u(x-a)\right] + \underline{E}\left[u(x-a) - u(x-d)\right] \tag{3}
$$

where $\underline{E}_0$ and $\underline{E}$ are the electric field phasors external and internal to the dielectric slab, respectively. Using the Gaussian boundary conditions at x=a (i.e., $\varepsilon\underline{E} - \varepsilon_0\underline{E}_0 = \underline{\sigma}_{sfa}$ and $\sigma\underline{E} = -j\omega\underline{\sigma}_{sfa}$ due to the presence of free surface charge density represented by the phasor $\underline{\sigma}_{sfa}$) and Faraday's law (i.e., $\underline{E}_0 a + \underline{E} t = V_0$) [6], we find:

$$
\underline{E}_0 = \frac{V_0}{d}\,\frac{1 + j\omega\varepsilon/\sigma}{\left[1 - t/d\right] + \left\{1 - (t/d)\left(1-\varepsilon_r^{-1}\right)\right\} j\omega\varepsilon/\sigma} \tag{4}
$$

$$
\underline{E} = \frac{V_0}{d}\,\frac{j\omega\varepsilon_0/\sigma}{\left[1 - t/d\right] + \left\{1 - (t/d)\left(1-\varepsilon_r^{-1}\right)\right\} j\omega\varepsilon/\sigma} \tag{5}
$$

Now, using Newton's third law, we can calculate the desired total ac force $\langle \underline{F}_x \rangle$ exerted on the lower dielectric-coated conductor plate by, instead, simply calculating the equal and opposite ac force on the upper conductor plate of the capacitor. This equivalent ac force on the upper plate can be calculated in one simple step by expressing the Coulomb force directly in terms of the new special integral identity (1). The total free surface charge density phasor $\underline{\sigma}_{sf0}$ induced on the x=0 surface of the upper conductor plate shown in Fig. 2 can be expressed as an infinitely thin equivalent volume charge density phasor with the use of the unit impulse function as $\underline{\rho}_v = \underline{\sigma}_{sf0}\delta(x)$ where $\underline{\sigma}_{sf0} = \varepsilon_0 \underline{E}_0$ due to the Gaussian boundary condition at x=0. Therefore, using the above expressions for $\underline{E}_x$ and $\underline{\rho}_v$, and the new technique described in this paper based on (1), the total

Coulomb ac force exerted on the lower dielectric-coated conductor plate is found by calculating the negative of the corresponding ac force exerted on the upper conductor plate as follows:

$$\begin{aligned}\langle F_x \rangle &= -\langle F_x|_{x=0} \rangle = -\left\langle \int_{V|_{x=0}} \rho_v E_x \, dv \right\rangle \\ &= -\frac{1}{2}\mathrm{Re}\left\{ \int_{V|_{x=0}} \rho_v E_x^* \, dv \right\} \\ &= -\frac{1}{2}\mathrm{Re}\left\{ A \int_{0^-}^{0^+} \underbrace{\sigma_{sf0}\delta(x)}_{\rho_v} \underbrace{E_0^* u(x)}_{E_x^*} dx \right\} \\ &= -\frac{1}{2}\mathrm{Re}\{\sigma_{sf0} E_0^*\} A \underbrace{\int_{0^-}^{0^+} \delta(x) u(x) dx}_{1/2} \\ &= -\frac{A}{4}\varepsilon_0 |E_0|^2 \end{aligned} \quad (6)$$

Note once again that (1) is the source of the essential *additional* factor of "1/2" in (6) and the sign of the ac force $\langle F_x \rangle$ is negative meaning that the direction of the force is upward (i.e., attractive as shown in Fig. 2) as expected.

After substituting for E_0 from (4) into (6), we get the final total analytical closed-form time-average Coulomb ac force as follows:

$$\langle F_x \rangle = \frac{\varepsilon_0 V_0^2 A}{4d^2} \frac{\tan^2\delta + 1}{(1 - t/d)^2 \tan^2\delta + \left[1 - (t/d)\left(1 - \varepsilon_r^{-1}\right)\right]^2} \quad (7)$$

where $\tan\delta = \sigma/(\omega\varepsilon)$ is the loss tangent of the dielectric-coating material. Note that we dropped the negative sign of the ac force expression for generality.

Let us now consider two interesting limiting cases for (7). In the first limiting case, assume $\omega \to 0$ (i.e., low-frequency source) and $\sigma \neq 0$. In this case, $\tan\delta = \sigma/(\omega\varepsilon) \to \infty$ and the ac force expression (7) reduces to:

$$\langle F_x \rangle = \frac{\varepsilon_0 V_0^2 A}{4(d - t)^2} \quad (8)$$

which is the expected Coulomb ac force exerted on each conductor plate of a vacuum-filled parallel-plate capacitor with separation gap $d-t$. Note that in this case, the dielectric material behaves like a perfect conductor and, therefore, (8) is independent of ε_r.

In the second limiting case, assume $\sigma=0$ (i.e., lossless dielectric slab) and $\omega \neq 0$. In this case, $\tan\delta = \sigma/(\omega\varepsilon) \to 0$ and the ac force expression (7) simplifies to:

$$\langle F_x \rangle = \frac{\varepsilon_0 V_0^2 A}{4} \frac{1}{\left[d - t\left(1 - \varepsilon_r^{-1}\right)\right]^2} \quad (9)$$

Note that the force is now a function of ε_r. If we let $\varepsilon_r \to \infty$ in (9), the ac force becomes equal to (8) as expected since, again, the dielectric material with $\varepsilon_r = \infty$ behaves like a perfect conductor. Alternatively, if we let $\varepsilon_r \to 1$ in (9), we find that $\langle F_x \rangle \to \varepsilon_0 V_0^2 A/(4d^2)$ since this is the special case where the dielectric slab effectively does not exist.

3 RESULTS AND DISCUSSION

In this section, we graphically present the results of (7) by plotting a family of curves showing the variation of the ac force as a function of both loss tangent ($\tan\delta$) and ε_r at a specific t/d ratio. Specifically, in Fig. 3, the *normalized* magnitude of the ac force given by $F_n = \langle F_x \rangle / [\varepsilon_0 V_0^2 A/(4d^2)]$ is plotted as a function of $\tan\delta$ (ranging from 10^{-3} to 10^3) for five different values of ε_r at the specific ratio t/d=0.5. Notice that the two asymptotic values of F_n in Fig. 3 reduce to the interesting limiting cases given by (8) and (9) at the two extreme values of the loss tangent. In addition, it is clearly seen that the force is dependent on ε_r for the limiting case when $\tan\delta \ll 1$ as expected from (9), whereas, when $\tan\delta \gg 1$, the force is constant and independent of ε_r as expected from (8). For example, in the case when ε_r=1 in Fig. 3, the two limiting values of F_n are four and one as expected from (8) and (9), respectively.

The physical nature of the Coulomb ac force on the dielectric-coated lower conductor plate $\langle F_x \rangle$ in (7) is due to the phenomenon of dielectric relaxation as well as the presence of both *free* and *polarization* surface charges at both the vacuum/dielectric and dielectric/conductor interfaces at $x=a$ and $x=d$, respectively. Therefore, the total Coulomb ac force exerted on the lower dielectric-coated conductor consists of the sum of two individual ac forces such that $\langle F_x \rangle = \langle F_{xa} \rangle + \langle F_{xd} \rangle$ where $\langle F_{xa} \rangle$ is the ac force exerted at the dielectric/vacuum interface at $x=a$ and, similarly, $\langle F_{xd} \rangle$ is the ac force at the surface of the lower conductor plate at $x=d$. Both $\langle F_{xa} \rangle$ and $\langle F_{xd} \rangle$ can be directly calculated independently using the new technique described in this paper. This detailed calculation is beyond the scope of this paper. If they are calculated separately, their ratio is found to be:

$$\langle F_{xa} \rangle / \langle F_{xd} \rangle = \varepsilon_r^2 \left(\tan^2\delta + 1\right) - 1 \quad (10)$$

Note that depending on the value of the loss tangent and ε_r, one of these individual forces may dominate the total ac force given by (7). We can conveniently re-express the loss tangent as $\tan\delta = (2\pi)^{-1} \mathrm{T}/(\varepsilon/\sigma)$ where $\mathrm{T} = 2\pi/\omega$ is the period of oscillation of the ac voltage source. Therefore, the loss tangent can be interpreted as a quantity which is directly proportional to the ratio of the period of oscillation T and the material's dielectric relaxation time ε/σ. In other words, the quantities $\mathrm{T}/(\varepsilon/\sigma)$ and ε_r determine the relative contributions of $\langle F_{xa} \rangle$ and $\langle F_{xd} \rangle$ to the total net ac force in (7). For example, for low frequencies (i.e., $\tan\delta \gg 1$), $\langle F_{xa} \rangle / \langle F_{xd} \rangle \approx \varepsilon_r^2 \tan^2\delta = \varepsilon_r^2 \left[(2\pi)^{-1} T / (\varepsilon/\sigma)\right]^2 >> 1$ and, therefore, $\langle F_{xa} \rangle$ is the major contributor to the net ac force. This is due to the ability of the free surface charge at $x=a$ to remain in phase with the low-frequency oscillations (i.e., $T \gg \varepsilon/\sigma$). Similarly, for high frequencies (i.e., $\tan\delta \ll 1$), $\langle F_{xa} \rangle / \langle F_{xd} \rangle \approx (\varepsilon_r^2 - 1)$ and, therefore, either $\langle F_{xa} \rangle$ or $\langle F_{xd} \rangle$ may be the major contributor to the net ac force depending on ε_r. This is due to the fact that the free surface charge at $x=a$ can now no longer remain in phase with the high-frequency oscillations (i.e., $T \ll \varepsilon/\sigma$).

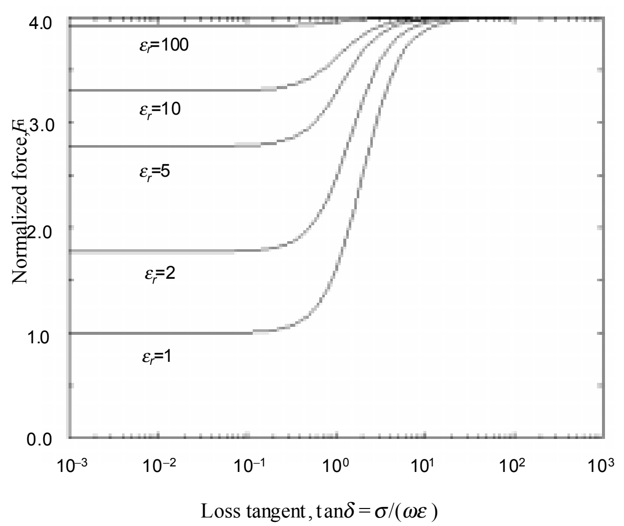

Figure 3: Plot of normalized force F_n versus both loss tangent and ε_r for t/d=0.5.

4 CONCLUSIONS

In this paper, we have utilized a new special singularity integral identity given by (1) to directly and straightforwardly calculate and plot the closed-form analytical expression for the time-average electroquasistatic sinusoidal steady-state Coulomb ac force exerted on a solid incompressible lossy dielectric slab attached to the lower conductor plate of a parallel-plate capacitor. The final results for the ac force clearly show the important and interesting dependence of the force on the material dielectric constant ε_r as well as the loss tangent ($\tan\delta$) which is set by the operating frequency ω of the driving ac voltage source and the material properties (σ and ε) of the dielectric material. In other words, it is the relative values of the period of oscillation of the ac voltage source $T=2\pi/\omega$ and the dielectric relaxation time ε/σ of the dielectric material in addition to ε_r which determine the magnitude of the final resulting net ac force provided in (7) and plotted in Fig. 3.

This example demonstrates the power, simplicity and utility of this new technique to directly calculate electrostatic and electroquasistatic Coulomb forces in analytical closed-form. The authors believe that this technique for calculating the ac force on a dielectric-coated conductor is of interest to the MEMS community in related problems involving MEMS structures actuated by electroquasistatic forces. For example, this technique could be used to extend the work of Osterberg, et al [7] to simulate electroquasistatic-induced beam bending in cantilever or bridge beam MEMS test structures under more complex conditions such as ac voltage drive and the presence of a lossy dielectric coating. Only pure highly-doped silicon MEMS beam structures suspended over a ground plane under dc voltage drive were considered in [7].

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