Air-damping Effect on a Micro- and Nano-machined Beam Resonator

G. Xu, C. Zhang and Q. Jiang

College of Engineering, University of California at Riverside
Riverside, CA 92521, gxu@engr.ucr.edu

ABSTRACT

The air-damping effect on the resonant frequency and the quality factor of a micro- and nano-machined beam resonator is studied. The beam, placed in a uniform magnetic field, is driven to vibrate transversely by the Lorentz force generated by an electrical signal passing through a lead attached to the beam. Based on the Oseen solution of the drag force acting on an infinite long cylinder that moves in incompressible viscous fluids at low Reynolds numbers, the air drag to the beam vibration is characterized and incorporated into the linear elastic beam theory. The analytical results show that air-damping generally shifts the resonant frequency downward and degrades the quality-factor, and that this effect increases as the dimension of the beam decreases. In addition, the frequency response of the electromotive force generated by the motion of the resonator in the magnetic field is also obtained, including the influence of high frequency modes. Based on the quantitative numerical results, it is concluded that the air-damping effect can be significant for sub-micron resonators for frequency-agile applications, while the high frequency mode effect appears to be negligible under realistic physical circumstances.

Keywords: Air-damping, Beam, Resonator, Resonant Frequency, Electromotive Force.

1 INTRODUCTION

Micro- and nano-machined mechanical resonators have been received considerable attention recently because of their many important technological applications. For example, such resonators with central frequencies in the order of MHz or even GHz, typically fabricated from the single crystal silicon [1, 2], have potential to be used as components of signal filters in communications systems [3]. Accurate analysis of various mechanical effects on the characteristics of resonators, such as resonant frequencies and quality factors, is essential for designing high performance components.

A primary concern of the performance of micro- and nano-machined mechanical resonators is the air-damping effect on the resonant frequency and the quality factor. The demand of high frequency application requires the decrease of the dimension of the resonator. However, because the volume (mass) decreases more rapidly than the surface area, this would increase the air-damping effect accordingly, resulting in the intolerable resonant frequency shift and the degradation of the quality factor. The analysis of this length-scale dependent air-damping effect will provide the guidelines to determine under what conditions the resonator needs to be vacuum-encapsulated.

In this communication, we present an analysis of the air-damping effect on the vibration of a double clamped elastic prism beam made from the bulk silicon crystal. The beam, placed in the uniform magnetic field, is driven to vibrate transversely by the Lorentz force generated by an electrical signal passing through a lead attached on the beam. A beam resonator of this type of configuration was fabricated and studied by Cleland and Roukes [1, 3]. Here, we shall emphasize on the air-damping effect on the resonant frequency shift and the quality factor of the resonator as the dimension of the resonator varies.

2 MECHANICAL ANALYSIS

Consider a double clamped elastic prism beam as shown in Fig. 1. The length, the height, and the width of the beam are l, h, and a, respectively. The magnetic field **B** is assumed to be normal to the beam and the paper. The input electric current $I_D(t)$ passes through the lead attached on the beam in the x-axis direction. The beam is activated to vibrate transversely in Y-axis direction by the Lorentz force.

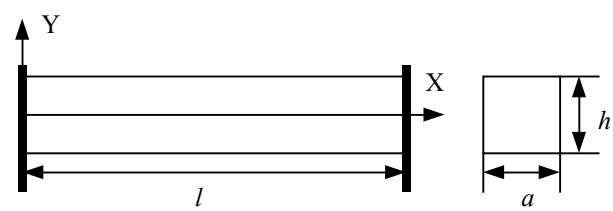

Fig. 1 Geometry of the double clamped elastic prim beam

2.1 Equation of Vibration

Following the continuum theory of the beam [4], the motion equation of the beam can be expressed as

$$\frac{Eah^3}{12}\frac{f^4 y(x,t)}{fx^4} + \rho ha\frac{f^2 y(x,t)}{ft^2} = q(x,t) \qquad (1)$$

where E is Young's modulus; ρ is the density of the beam; $q(x, t)$ is the external loading per unit length on the beam, which consists of air drag q_d and the Lorentz force q_m generated by the electric current in the magnetic field.

To determine the drag q_d, we resort to the definition of the drag force F_d on the rigid body immersed in the incompressible fluid flow with the speed V [5],

$$F_d = \frac{1}{2}\rho_a V^2 C_d A \tag{2}$$

where ρ_a is the density of air, A is the wetted area, C_d is the drag coefficient, depending on the Reynolds numbers. To obtain the drag force on the beam considered, we begin by considering an infinite long cylinder that moves with the speed V in the air, with the Reynolds number less than 1.0, which appears to be a valid assumption for the micro- and nano-machined beam vibration. The Oseen solution for the drag is given by Tomotika and Aoi [6, 7]

$$C_d = \frac{2F_d^{/}}{\rho_a V^2 d} = \frac{8\pi}{\mathrm{Re}[0.5 - \tilde{A} + \ln(8/\mathrm{Re})]} \tag{3}$$

where $F_d^{/}$ is the drag per unit length; d is the diameter of the cylinder; Γ = 0.577216… is the Euler's constant; Re is the Reynolds number defined by

$$\mathrm{Re} = \frac{\rho_a V d}{\mu} \tag{4}$$

where μ is the dynamic viscosity of air. The equation (3) is in good agreement with the experimental result for Re < 1 [5]. Define

$$\alpha = \frac{4\pi}{0.5 - \tilde{A} + \ln(8/\mathrm{Re})}, \tag{5}$$

the drag force per unit length on the cylinder may be expressed as

$$q_d = \alpha \mu V . \tag{6}$$

We assume that the form of equation (6) is also applicable to the long prism beam with the modified α by a coefficient. The value of the coefficient may be calibrated from the ratio of the drag coefficient of the beam to that of the cylinder at the high Reynolds numbers, which is given by about 1.67 [8]. Therefore we express the air drag force per unit length on the beam by

$$q_d = \alpha \mu \frac{dy(x,t)}{dt} . \tag{7}$$

Substitution of (7) into (1) would result in a non-linear equation that requires numerical results. Furthermore, we note that α changes flatly with the Reynolds because of the logarithm term. Hence, we simplify the equation (1) by assuming the parameter α to be a constant. The final air-damping effect will be estimated by selecting the parameter α in the range of 0 to 10, corresponding to the appropriate range of the Reynolds number for the analysis. Finally, the Lorentz force in (1) is given by

$$q_m = BI_D(t) \tag{8}$$

where B is the intensity of magnetic field, I_D is the input electric current. Substitution of (7) and (8) into (1) results in the motion equation of the beam

$$\frac{f^4 y}{fx^4} + \frac{12\rho}{Eh^2}\frac{f^2 y}{ft^2} + \frac{12\alpha\mu}{Eh^3 a}\frac{fy}{ft} = \frac{12BI_D}{Eah^3} \tag{9}$$

with the boundary conditions

$$y(0,t) = y''(0,t) = 0, \qquad y(l,t) = y''(l,t) = 0 \tag{10}$$

1.2 Mode Analysis of Free Vibration

We first consider the free vibration, in which there is no exciting magnetic force term. By separation of variables and use of the boundary conditions, we obtain the solution of free vibration

$$y(x,t) = \sum_i e^{-\zeta_i p_i t}\left(A_i \sin\sqrt{1-\zeta_i^2}\, p_i t + B_i \cos\sqrt{1-\zeta_i^2}\, p_i t\right) X_i(x) \tag{11}$$

where A_i and B_i are coefficients determined by initial conditions, and

$$X_i = \mathrm{ch}\beta_i x - \cos\beta_i x + \gamma_i(\mathrm{sh}\beta_i x - \sin\beta_i x) \tag{12}$$

where

$$\gamma_i = -\frac{\mathrm{ch}\beta_i l - \cos\beta_i l}{\mathrm{sh}\beta_i l - \sin\beta_i l}, \tag{13}$$

$$\beta_i l = \bar{\beta}_i, \quad \bar{\beta}_1 = 4.73, \quad \bar{\beta}_i \cup (i+0.5)\pi (i = 2,3...), \tag{14}$$

$$p_i = \frac{\bar{\beta}_i^2 h}{l^2}\sqrt{\frac{E}{12\rho}}, \tag{15}$$

and

$$\zeta_i = \frac{\alpha\mu l^2}{ah^2\bar{\beta}_i^2}\sqrt{\frac{3}{E\rho}} \tag{16}$$

where the parameter p_i is the natural frequency of the beam free of air damping; ζ_i is conventionally defined as the damping ratio of the dynamical system.

The above solution gives the resonant frequency

$$\omega_{ri} = \sqrt{1-2\zeta_i^2}\, p_i . \tag{17}$$

The resonant frequency shift ratio is then given by

$$\frac{\Delta\omega}{\omega} = \frac{p_i - \omega_{ri}}{p_i} = 1 - \sqrt{1-2\zeta_i^2} . \tag{18}$$

The solution of the first mode is of the most interest. For the small dumping ratio, the quality factor Q, for the first mode vibration, can be approximately expressed by

$$Q = \frac{1}{2\zeta_1} . \tag{19}$$

To illustrate the size dependent air-damping effect on the natural frequency, resonant frequency shift, and the quality factor, we consider a fixed shape (l/a=20, h/a=2) beam with various width. The shape of beam resembles the silicon beam with length of 7.7 μm, width of 0.33 μm, and

height of 0.8 μm, fabricated by Cleland and Roukes [1]. Only effect on the first mode vibration is considered. The materials constants are taken $E = 1.658 \times 10^{11}$ N/m^2, ρ = 2300 kg/m^3, $\mu = 1.81 \times 10^{-5}$ Ns/m^2. Fig. 2 demonstrates the

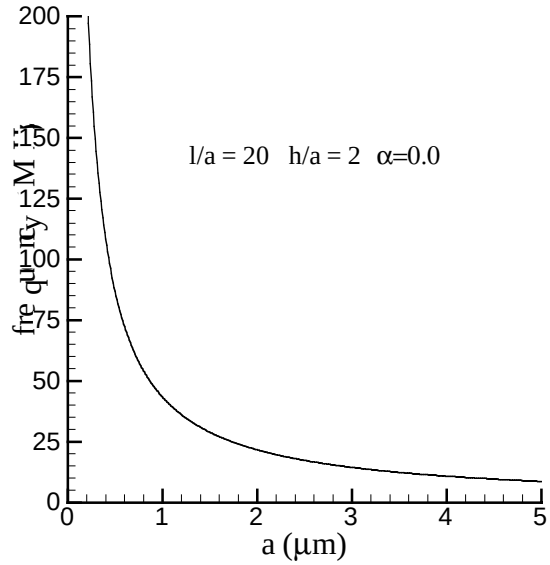

Fig. 2 Variation of the resonant frequency with the width of the fixed shaped beam

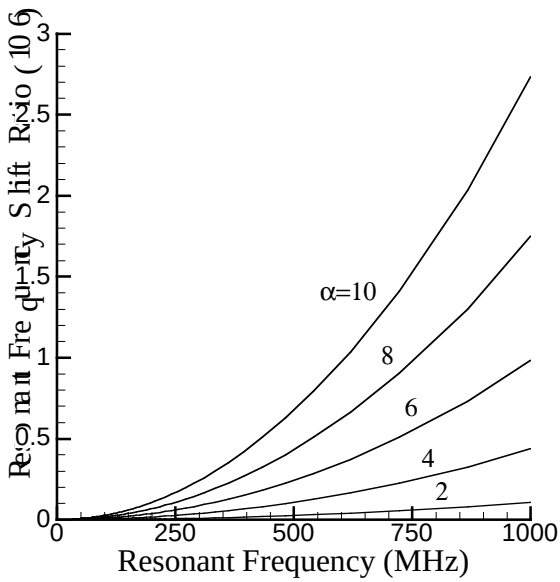

Fig. 3 Variation of the resonant frequency shift ratio with the resonant frequency for various damping coefficients

dependence of the natural frequency on the width of the beam. It shows the general trend that the frequency increases as the dimension of the beam decreases. Fig. 3 illustrates the variation of resonant frequency shift ratio with the resonant frequency for various parameter α. It shows the resonant frequency shift ratio increases as the resonant frequency increases. This is because the higher frequency corresponds to the smaller beam, in which the air-damping effect is relatively more dominant. By the same token, the quality factor decreases as the frequency increases, which is shown in Fig 4.

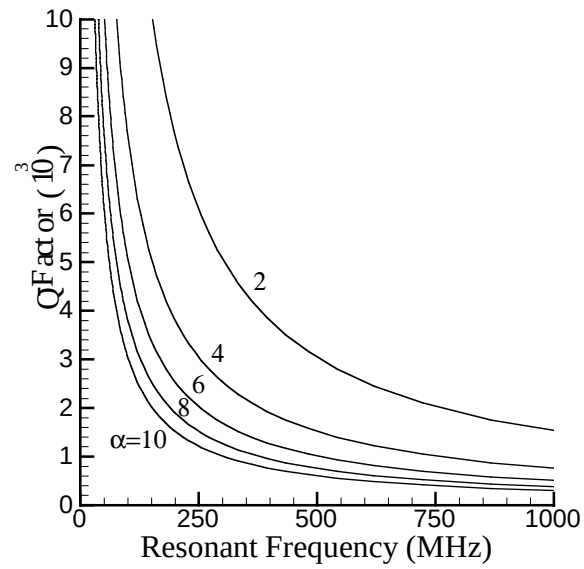

Fig. 4 Variation of the Q-factor with the resonant frequency for various damping coefficients

1.3 Solution of Forced Vibration

To obtain the solution of the forced vibration under the magnetic force, we assume $I_D(t) = Ie^{j\omega t}$. By expanding the solution of (9) in the vibrational modes X_i and using the orthogonal character of the modes, we can obtain the solution

$$y(x,t) = \sum_i \frac{BI\varphi_i}{ah\rho} \frac{X_i(x)e^{j\omega t}}{p_i^2 - \omega^2 + j2\zeta_i p_i \omega} \tag{20}$$

where

$$\varphi_i = \frac{\int_0^l X_i(x)dx}{\int_0^l X_i^2(x)dx} \tag{21}$$

which has non-zero odd terms $\varphi_1 = 0.831$, $\varphi_3 = 0.364$, $\varphi_5 = 0.231$.

The motion of the resonator through the magnetic field generates an electromotive force (EMF) along the leads attached to the resonator. In the absence of an external electrical circuit, or, equivalently, one with an infinite source impedance, the voltage is given by

$$V_{EMF} = \int_0^l B \frac{fy}{ft} dx \ . \tag{22}$$

Substituting (20) into (22), we obtain

$$V_{EMF} = \sum_i \frac{jlIB^2\omega}{ah\rho} \frac{\varphi_i^{\,2} e^{j\omega t}}{p_i^{\,2} - \omega^2 + j2\zeta_i p_i \omega}. \qquad (23)$$

The above exact solution includes the influence of the higher modes of vibration. As a remark, we note that the solution also differs from the solution given by Cleland and Roukes [2] by the factor of φ_1 if only the basic mode of vibration is considered. This difference is caused by the fact that the "effective'' concept used in their analysis was not applied on the both side of equation (1) in their paper.

By selecting a = 1 μm, h = 0.5 μm, l = 20 μm, α = 10, we have studied the effect of the higher mode vibration on the first mode vibration. It shows that the maximum magnitude of the EMF and qualify factor remain almost the identical. We therefore conclude that the effect of higher mode on the basic mode can be neglected in realistic physical applications.

2 SUMMARY AND CONCLUSION

The air-damping effect on the resonance frequency and the quality factor of a beam resonator is studied. Based on the Oseen solution of the drag force acting on an infinite long cylinder that moves in an incompressible viscous fluid with uniform velocity at low Reynolds numbers, the air drag to the beam vibration is characterized and incorporated into the continuum beam theory. The analytical results have shown that air-damping generally shifts the resonance frequency downward and degrades the quality-factor, and that this effect increases as the dimension of the beam decreases. In addition, the frequency response of the electromotive force generated by the motion of the resonator in the magnetic field is also obtained, including the influence of high frequency modes. Based on the quantitative numerical results, it is concluded that the air-damping effect can be significant for sub-micron resonators for frequency-agile applications, while the high frequency mode effect appears to be negligible under realistic physical circumstances.

ACKNOWLEDGEMENT

GX acknowledges the support of the Regents Faculty development award. CZ acknowledges the support of the China National Petroleum Corporation fellowship. QJ acknowledges the support of the US Office of Naval Research through Grant No. N00014-96-1-0884, the US National Science Foundation through Grant No. MSS-9308341 and the California Space Institute through Grant No. CS-43-98.

REFERENCES

[1] A. N. Cleland and M. L. Roukes, "Fabrication of high frequency nanometer scale mechanical resonators from bulk Si crystals", Appl. Phys. Lett., Vol. 69, No.18, 2653-2655, 1996

[2] A. N. Cleland and M. L. Roukes, "External control of dissipation in a nanometer-scale radiofrequency mechanical resonator", Sensors and Actuators, 72, 256-261, 1999.

[3] C. T.-C. Nguyen and R. T. Howe, "An Integrated CMOS Micromechanical Resonator High-Q Oscillator", IEEE Journal of Solid-State Circuits, Vol. 34, No. 4, 440-455, 1999.

[4] S. P. Timoshenko and W. Weaver, Vibration Problems in Engineering, 4th ed., John Wiley, New York, 1974.

[5] R. W. Fox and A. T. McDonald, Introduction to Fluid Mechanics, 5th ed. John Wiley and Sons, New York, 1999.

[6] F. M. White, Viscous Fluid Flow, 2nd, ed., McGraw-Hill, New York, 1991.

[7] S. Tomotika and T. Aoi, "An expansion formula for the drag on a circular cylinder moving through a viscous fluid at small Reynolds number", Quart. J. Mech. Appl. Math., Vol.4, 401-406, 1951.

[8] H. Schlichting, Boundary Layer Theory, 7th McGraw-Hill, New York, 1979.